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Second order approximation for variance of SEP

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SECOND ORDER APPROXIMATION FOR
VARIANCE OF SEP

by

Arthur Frederick Brock

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Second Order Approximation For Variance of SEP

by

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Lieutenant, United States Navy
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ABSTRACT

A method for the computation of confidence intervals for circular error probability (CEP) based on first order variance estimates was introduced in 1966. It was later found that under certain conditions the resulting confidence intervals for CEP were smaller than expected. As a result a second order variance estimate method was developed, at the Johns Hopkins University Applied Physics Laboratory, which greatly improved the accuracy of the confidence intervals for CEP. The purpose of this thesis is to develop and test procedures for the 3-dimensional case to obtain a second order estimate for variance of spherical error probability (SEP).

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The reader is cautioned that computer programs developed in this research may not have been exercised for all cases of interest. While every effort has been made, within the time available, to ensure that the programs are free of computational and logic errors, they cannot be considered validated. Any application of these programs without additional verification is at the risk of the user.

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I. INTRODUCTION

A. BACKGROUND

In 1966 W.R. Blischke and A.H. Halpin [Ref. 1] first described methods for the computation of confidence intervals for circular error probability (CEP) based on first order variance estimates. When these procedures were implemented at the Johns Hopkins University Applied Physics Laboratory (JHU-APL) it was found that under certain conditions the resulting confidence intervals for CEP were smaller than expected. As a result R.C. Ferguson, K.V. Kitzman and P.B. Jackson [Ref. 2] have extended the Blischke-Halpin procedures to create a second order variance estimate. Testing conducted by Kitzman [Ref. 3] has confirmed that this method produces a more accurate estimate for the variance of CEP.

Our goal will be to extend this procedure to the 3-dimensional case to obtain a second order estimate for variance of spherical error probability (SEP).

B. METHODOLOGY

We begin by assuming that missile detonations are distributed as trivariate Gaussian. In the past, analysis of ballistic missile test firing data has failed to disprove this assumption. If the mean and variance are given by:

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}.$$

Then the probability of a detonation occurring within a distance R of the origin is given by:

$$P(R;\mu,\Sigma) = \iiint_D \frac{1}{(\sqrt{2\pi})^3 q} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\mu)^T \Sigma^{-1} (\boldsymbol{x}-\mu)\right\} dx_1 dx_2 dx_3$$

where,

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

$$q^2 = \det[\Sigma],$$

and $D = \left\{(x_1, x_2, x_3) \middle| x_1^2 + x_2^2 + x_3^2 \le R^2\right\}.$

By the implicit function theorem, $\frac{\partial P}{\partial R} > 0$ implies that the equation $P(R;\mu,\Sigma)$ = CONSTANT defines a differentiable function

$R(\mu, \Sigma)$ such that $P(R(\mu, \Sigma); \mu, \Sigma)$ = CONSTANT. If the CONSTANT is set to $\frac{1}{2}$, then $R(\mu, \Sigma)$ is the SEP function. Let $X_1, X_2, \ldots, X_n$ be a random sample of independent and identically distributed random vectors with distribution $N(\mu, \Sigma)$. Then the maximum likelihood estimators of μ and Σ are $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\hat{\Sigma} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^T(X_i - \hat{\mu})$. When $\hat{\mu}$ and $\hat{\Sigma}$ replace μ and Σ in the equation $P(R; \mu, \Sigma) = \frac{1}{2}$ then $R(\hat{\mu}, \hat{\Sigma})$ gives $S\hat{E}P$, an estimator of SEP.

The second-order approximation for the variance of $S\hat{E}P$ is given by:

$$\sigma^2_{S\hat{E}P} = \left(D_{\mu\Sigma^u} SEP\right)^T P \left(D_{\mu\Sigma^u} SEP\right)$$

$$+ \frac{1}{2}\left(\underline{D^2_{\mu\Sigma^u} SEP}\right)^T (P \otimes P)\left(\underline{D^2_{\mu\Sigma^u} SEP}\right)$$

$$+ \frac{1}{4}\left[\left(\underline{D^2_{\mu\Sigma^u} SEP}\right)^T \underline{P}\right]^2 \tag{1}$$

where $\Sigma^u = (\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33})^T$, $D_{\mu,\Sigma^u} SEP$ is the derivative of SEP with respect to μ, Σ^u viewed as a column vector, $D^2_{\mu\Sigma^u} SEP$ is the usual second derivative matrix (or Hessian) of SEP with respect to μ, Σ^u, $P = \begin{pmatrix} P_\mu & P_{\mu\Sigma^u} \\ P^T_{\mu\Sigma^u} & P_{\Sigma^u} \end{pmatrix}$ is the variance-covariance matrix $\hat{\mu}$, $\hat{\Sigma}^u$, $\otimes$ is the tensor product operation and the underline notation is used to represent the column vector formed by stringing out the elements of the matrix by rows. For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $\underline{A} = [1\ \ 2\ \ 3\ \ 4]^T$. Note that $\hat{\mu}$ is distributed $N(\mu, \frac{1}{n}\Sigma)$ so $P_\mu = \frac{\Sigma}{n}$. It can also be seen that $n\hat{\Sigma}$ has a Wishart distribution with parameter Σ, so that $P_\Sigma = \frac{2}{n}(\Sigma \otimes \Sigma)$. The complete derivation of equation (1), based on Taylor Series expansion, is given in appendix A.

Since $\hat{\mu}$ and $\hat{\Sigma}$ are independent [Ref. 4:p. 102] it follows that $P_{\mu,\Sigma^u} = 0$ and equation (1) can be written as:

$$\sigma^2_{S\hat{E}P} = \left(\frac{\partial SEP}{\partial \mu}\right)^T P_\mu \left(\frac{\partial SEP}{\partial \mu}\right) + \left(\frac{\partial SEP}{\partial \Sigma^u}\right)^T P_{\Sigma^u} \left(\frac{\partial SEP}{\partial \Sigma^u}\right)$$

$$+\frac{1}{2}\left[\left(\frac{\partial^2 SEP}{\partial \mu^2}\right)^T (P_\mu \otimes P_\mu)\left(\frac{\partial^2 SEP}{\partial \mu^2}\right) + \left(\frac{\partial^2 SEP}{\partial \Sigma^{u^2}}\right)^T (P_{\Sigma^u} \otimes P_{\Sigma^u})\left(\frac{\partial^2 SEP}{\partial \Sigma^{u^2}}\right)\right]$$

$$+\frac{1}{2}\left[\left(\frac{\partial^2 SEP}{\partial \Sigma^u \partial \mu}\right)^T (P_\mu \otimes P_{\Sigma^u})\left(\frac{\partial^2 SEP}{\partial \Sigma^u \partial \mu}\right) + \left(\frac{\partial^2 SEP}{\partial \mu \partial \Sigma^u}\right)^T (P_{\Sigma^u} \otimes P_\mu)\left(\frac{\partial^2 SEP}{\partial \mu \partial \Sigma^u}\right)\right]$$

$$+\frac{1}{4}\left(\left(\frac{\partial^2 SEP}{\partial \mu^2}\right)^T (\underline{P_\mu}) + \left(\frac{\partial^2 SEP}{\partial \Sigma^{u^2}}\right)^T (\underline{P_{\Sigma^u}})\right)^2 . \tag{2}$$

Derivation of the first order terms in equation (2) has been accomplished by S.D. Hill at JHU-APL [Ref. 5]. In order to solve for the second order terms it will be necessary to find expressions for $\frac{\partial^2 SEP}{\partial \mu^2}$, $\frac{\partial^2 SEP}{\partial \Sigma^{u^2}}$ and $\frac{\partial^2 SEP}{\partial \mu \partial \Sigma^u}$, i.e. it is necessary to find $D^2_{\mu,\Sigma^u} SEP$, the second derivative matrix of SEP with respect to its arguments. Chapter II will describe the derivation of the second order variance estimate and Chapter III will discuss implementation and testing of the algorithm. Conclusions will be presented in Chapter IV.

II. DERIVATION OF THE SECOND ORDER ESTIMATE

A. FIRST ORDER TERMS

We begin by describing the derivation of the first order terms given in equation (2). If we define:

$$F(x_1, x_2, x_3; \mu, \Sigma^u) = \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\mu)^T\Sigma^{-1}(\boldsymbol{x}-\mu)\right\}$$

where

$$\Sigma^{-1} = \begin{bmatrix} \sigma^{11} & \sigma^{12} & \sigma^{13} \\ \sigma^{21} & \sigma^{22} & \sigma^{23} \\ \sigma^{31} & \sigma^{32} & \sigma^{33} \end{bmatrix}, \quad \begin{matrix} \sigma^{12} = \sigma^{21} \\ \sigma^{13} = \sigma^{31} \\ \sigma^{23} = \sigma^{32} \end{matrix}$$

$$\begin{aligned}(\boldsymbol{x}-\mu)^T\Sigma^{-1}(\boldsymbol{x}-\mu) = {} & (x_1-\mu_1)^2\sigma^{11} + 2(x_1-\mu_1)(x_2-\mu_2)\sigma^{21} + 2(x_1-\mu_1)(x_3-\mu_3)\sigma^{31} \\ & + (x_2-\mu_2)^2\sigma^{22} + 2(x_2-\mu_2)(x_3-\mu_3)\sigma^{23} + (x_3-\mu_3)^2\sigma^{33}\end{aligned}$$

and

$$g(\rho, \theta, \phi; \mu, \Sigma^u) = F(\rho\sin\theta\cos\phi, \rho\sin\theta\sin\phi, \rho\cos\theta; \mu, \Sigma^u).$$

Then, with a shift to spherical coordinates, we see that:

$$P(R; \mu, \Sigma^u) = \frac{1}{(\sqrt{2\pi})^3 q}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\theta\, g(\rho, \theta, \phi; \mu, \Sigma^u)\, d\rho\, d\theta\, d\phi. \tag{3}$$

We wish to solve equation (3) for each of the partials $\frac{\partial SEP}{\partial \lambda_i}$ where $\lambda = (\mu_1, \mu_2, \mu_3, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33})$. From the previous section, $R(\mu, \Sigma) = \frac{1}{2}$ is the SEP function, so we need to solve for the partials $\frac{\partial R}{\partial \lambda_i}$.

Taking partials of equation (3) we see that,

$$P'(R)\frac{\partial R}{\partial \lambda_i} = -\frac{1}{(\sqrt{2\pi})^3 Q}\iiint_D \frac{\partial F}{\partial \lambda_i} dx_1 dx_2 dx_3 \tag{4}$$

and

$$P'(R) = \frac{R^2}{(\sqrt{2\pi})^3 Q}\int_0^{2\pi}\int_0^{\pi} \sin\theta g(R, \theta, \phi) d\theta d\phi .$$

We can solve the right hand side of equation (4) for i=1,2,3 by using the relationships

$$\frac{\partial g}{\partial \mu_1} = -\frac{\partial g}{\partial x_1} \qquad \frac{\partial g}{\partial \mu_2} = -\frac{\partial g}{\partial x_2} \qquad \frac{\partial g}{\partial \mu_3} = -\frac{\partial g}{\partial x_3}$$

so we see that

$$-\iiint_D \frac{\partial g}{\partial \lambda_i} dx_1 dx_2 dx_3 = \iiint_D \frac{\partial g}{\partial x_i} dx_1 dx_2 dx_3$$

If we apply Green's Theorem to this equation and then make a shift to spherical coordinates we get:

$$-\iiint_D \frac{\partial g}{\partial \lambda_1} dx_1 dx_2 dx_3 = R^2 \int_0^{2\pi} \int_0^{\pi} \cos\phi \sin^2\theta\, g(R,\theta,\phi)\, d\theta\, d\phi$$

$$-\iiint_D \frac{\partial g}{\partial \lambda_2} dx_1 dx_2 dx_3 = R^2 \int_0^{2\pi} \int_0^{\pi} \sin\phi \sin^2\theta\, g(R,\theta,\phi)\, d\theta\, d\phi$$

$$\begin{aligned}-\iiint_D \frac{\partial g}{\partial \lambda_3} dx_1 dx_2 dx_3 &= R^2 \int_0^{2\pi} \int_0^{\pi} \cos\theta \sin\theta\, g(R,\theta,\phi)\, d\theta\, d\phi \\ &= \frac{R^2}{2} \int_0^{2\pi} \int_0^{\pi} \sin 2\theta\, g(R,\theta,\phi)\, d\theta\, d\phi\end{aligned}$$

To solve the right hand side of equation (4) for i=4,...,9 we first note that:

$$\frac{\partial^2 g}{\partial \mu_1 \partial \mu_2} = \begin{cases} 2\dfrac{\partial g}{\partial \sigma_{ij}}, & i=j \\ \dfrac{\partial g}{\partial \sigma_{ij}}, & i \neq j \end{cases}$$

so we get

$$\frac{\partial g}{\partial \sigma_{ij}} = \begin{cases} -\frac{1}{2}\frac{\partial}{\partial x_j}\frac{\partial g}{\partial \mu_i}\,, & i=j \\ -\frac{\partial}{\partial x_j}\frac{\partial g}{\partial \mu_i}\,, & i\neq j \end{cases}$$

which is used to evaluate the integral of $\frac{\partial g}{\partial \lambda_i}$ as follows:

$$-\iiint_D \frac{\partial g}{\partial \lambda_4}dx_1dx_2dx_3 = \frac{R^2}{2}\int_0^{2\pi}\int_0^{\pi}[\sigma^{11}(\rho\sin\theta\cos\phi-\mu_1)+\sigma^{12}(\rho\sin\theta\sin\phi-\mu_2)$$
$$+\sigma^{13}(\rho\cos\theta-\mu_3)]\cos\phi\sin^2\theta g(R,\theta,\phi)d\theta d\phi$$

$$-\iiint_D \frac{\partial g}{\partial \lambda_5}dx_1dx_2dx_3 = R^2\int_0^{2\pi}\int_0^{\pi}[\sigma^{11}(\rho\sin\theta\cos\phi-\mu_1)+\sigma^{12}(\rho\sin\theta\sin\phi-\mu_2)$$
$$+\sigma^{13}(\rho\cos\theta-\mu_3)]\sin\phi\sin^2\theta g(R,\theta,\phi)d\theta d\phi$$

$$-\iiint_D \frac{\partial g}{\partial \lambda_6}dx_1dx_2dx_3 = \frac{R^2}{2}\int_0^{2\pi}\int_0^{\pi}[\sigma^{11}(\rho\sin\theta\cos\phi-\mu_1)+\sigma^{12}(\rho\sin\theta\sin\phi-\mu_2)$$
$$+\sigma^{13}(\rho\cos\theta-\mu_3)]\sin2\theta g(R,\theta,\phi)d\theta d\phi$$

$$-\iiint_D \frac{\partial g}{\partial \lambda_7}dx_1dx_2dx_3 = \frac{R^2}{2}\int_0^{2\pi}\int_0^{\pi}[\sigma^{21}(\rho\sin\theta\cos\phi-\mu_1)+\sigma^{22}(\rho\sin\theta\sin\phi-\mu_2)$$
$$+\sigma^{23}(\rho\cos\theta-\mu_3)]\sin\phi\sin^2\theta g(R,\theta,\phi)d\theta d\phi$$

$$-\iiint_D \frac{\partial g}{\partial \lambda_8}dx_1dx_2dx_3 = \frac{R^2}{2}\int_0^{2\pi}\int_0^{\pi}[\sigma^{21}(\rho\sin\theta\cos\phi-\mu_1)+\sigma^{22}(\rho\sin\theta\sin\phi-\mu_2)$$
$$+\sigma^{23}(\rho\cos\theta-\mu_3)]\sin2\theta g(R,\theta,\phi)d\theta d\phi$$

$$-\iiint_D \frac{\partial g}{\partial \lambda_9} dx_1 dx_2 dx_3 = \frac{R^2}{4}\int_0^{2\pi}\int_0^{\pi}[\sigma^{31}(\rho\sin\theta\cos\phi-\mu_1)+\sigma^{32}(\rho\sin\theta\sin\phi-\mu_2)$$
$$+\sigma^{33}(\rho\cos\theta-\mu_3)]\sin 2\theta g(R,\theta,\phi)d\theta d\phi$$

If we define the functions:

$$CC_{i,j,k,l}(R;\mu,\Sigma^u) = \int_0^{2\pi}\int_0^{\pi}\cos^i(j\theta)\cos^k(l\phi)g(R,\theta,\phi)d\theta d\phi$$

$$CS_{i,j,k,l}(R;\mu,\Sigma^u) = \int_0^{2\pi}\int_0^{\pi}\cos^i(j\theta)\sin^k(l\phi)g(R,\theta,\phi)d\theta d\phi$$

$$SC_{i,j,k,l}(R;\mu,\Sigma^u) = \int_0^{2\pi}\int_0^{\pi}\sin^i(j\theta)\cos^k(l\phi)g(R,\theta,\phi)d\theta d\phi$$

$$SS_{i,j,k,l}(R;\mu,\Sigma^u) = \int_0^{2\pi}\int_0^{\pi}\sin^i(j\theta)\sin^k(l\phi)g(R,\theta,\phi)d\theta d\phi$$

Extending the work of Blischke and Halpin, with the aid of Green's Theorem, we can see that:

$$\frac{\partial SEP(\mu,\Sigma^u)}{\partial\mu} = \left[\frac{SC_{2,1,1,1}}{SC_{1,1,0,0}} \quad \frac{SS_{2,1,1,1}}{SC_{1,1,0,0}} \quad \frac{SC_{1,2,0,0}}{2SC_{1,1,0,0}}\right]^T_{\Big|(R(\mu,\Sigma^u);\mu,\Sigma^u)} \qquad (5)$$

$$\frac{\partial SEP(\mu,\Sigma^u)}{\partial \Sigma^u} = \frac{1}{2SC_{1,1,0,0}}\left.\begin{bmatrix} \dfrac{\partial SC_{2,1,1,1}}{\partial \mu_1} \\ 2\dfrac{\partial SS_{2,1,1,1}}{\partial \mu_1} \\ \dfrac{\partial SC_{1,2,0,0}}{\partial \mu_1} \\ \dfrac{\partial SS_{2,1,1,1}}{\partial \mu_2} \\ \dfrac{\partial SC_{1,2,0,0}}{\partial \mu_2} \\ \dfrac{\partial SC_{1,2,0,0}}{2\partial \mu_3} \end{bmatrix}\right|_{(R(\mu,\Sigma^u);\mu,\Sigma^u)}. \tag{6}$$

Applying the partials given in equation (6) to the partials $-\iiint_D \frac{\partial g}{\partial \lambda_i} dx_1 dx_2 dx_3$ given above we see that:

$$\frac{\partial SEP}{\partial \sigma_{11}} = \frac{1}{2w_5}\left\{\frac{R}{4}\left[4\sigma^{11}w_9 + 2\sigma^{12}w_{11} + \sigma^{13}(w_1 - w_2)\right] - w_8\left(\sigma^{11}\mu_1 + \sigma^{12}\mu_2 + \sigma^{13}\mu_3\right)\right\}$$

$$\frac{\partial SEP}{\partial \sigma_{12}} = \frac{1}{w_5}\left\{\frac{R}{4}\left[2\sigma^{11}w_{11} + 4\sigma^{12}w_{12} + \sigma^{13}(w_3 - w_4)\right] - w_{10}\left(\sigma^{11}\mu_1 + \sigma^{12}\mu_2 + \sigma^{13}\mu_3\right)\right\}$$

$$\frac{\partial SEP}{\partial \sigma_{13}} = \frac{1}{2w_5}\left\{\frac{R}{2}\left[\sigma^{11}(w_1 - w_2) + \sigma^{12}(w_3 - w_4) + \sigma^{13}(w_7 + w_5)\right] - w_6\left(\sigma^{11}\mu_1 + \sigma^{12}\mu_2 + \sigma^{13}\mu_3\right)\right\}$$

$$\frac{\partial SEP}{\partial \sigma_{22}} = \frac{1}{2w_5}\left\{\frac{R}{4}\left[2\sigma^{12}w_{11}+4\sigma^{22}w_{12}+\sigma^{23}(w_3-w_4)\right]\right.$$
$$\left.-w_{10}(\sigma^{12}\mu_1+\sigma^{22}\mu_2+\sigma^{23}\mu_3)\right\}$$

$$\frac{\partial SEP}{\partial \sigma_{23}} = \frac{1}{2w_5}\left\{\frac{R}{2}\left[\sigma^{12}(w_1-w_2)+\sigma^{22}(w_3-w_4)+\sigma^{23}(w_7+w_5)\right]\right.$$
$$\left.-w_6(\sigma^{12}\mu_1+\sigma^{22}\mu_2+\sigma^{23}\mu_3)\right\}$$

$$\frac{\partial SEP}{\partial \sigma_{33}} = \frac{1}{4w_5}\left\{\frac{R}{2}\left[\sigma^{13}(w_1-w_2)+\sigma^{23}(w_3-w_4)+\sigma^{33}(w_7+w_5)\right]\right.$$
$$\left.-w_6(\sigma^{13}\mu_1+\sigma^{23}\mu_2+\sigma^{33}\mu_3)\right\}$$

where $W(R;\mu,\Sigma^u)$ is given by:

$$W(R;\mu,\Sigma^u) = \begin{cases} w_1 = CC_{1,1,1,1} \\ w_2 = CC_{1,3,1,1} \\ w_3 = CS_{1,1,1,1} \\ w_4 = CS_{1,3,1,1} \\ w_5 = SC_{1,1,0,0} \\ w_6 = SC_{1,2,0,0} \\ w_7 = SC_{1,3,0,0} \\ w_8 = SC_{2,1,1,1} \\ w_9 = SC_{3,1,2,1} \\ w_{10} = SS_{2,1,1,1} \\ w_{11} = SS_{3,1,1,2} \\ w_{12} = SS_{3,1,2,1} \end{cases}$$

where each of the functions is evaluated at $R(\mu,\Sigma^u),\mu,\Sigma^u$.

This completes the derivation of the first order terms required in the computation of the variance estimate.

B. CONCEPT FOR DERIVATION OF THE SECOND ORDER TERMS

We first observe that equations (5) and (6) enable one to express $D^2_{\mu,\Sigma^u}SEP$ as a composition of three functions T_1, T_2 and T_3, each of which is a fairly simple function of its arguments.

Explicitly,

$$D_{\mu,\Sigma^u}SEP(\mu,\Sigma^u) = T_3 \circ T_2 \circ T_1(\mu,\Sigma^u)$$

where,

$$T_1(\mu,\Sigma^u) = \begin{bmatrix} \mu \\ \Sigma^u \\ SEP(\mu,\Sigma^u) \end{bmatrix}$$

$$T_2(\mu,\Sigma^u,R) = \begin{bmatrix} \mu \\ \Sigma^u \\ R \\ W(R;\mu,\Sigma^u) \end{bmatrix}$$

$$
T_3\begin{pmatrix}\mu\\ \Sigma^u\\ R\\ W\end{pmatrix} = \begin{bmatrix}
\frac{w_8}{w_5}\\
\frac{w_{10}}{w_5}\\
\frac{w_6}{2w_5}\\
\frac{R}{8w_5}\left(4\sigma^{11}w_9+2\sigma^{12}w_{11}+\sigma^{13}(w_1-w_2)\right)-\frac{w_8}{2w_5}\left(\sigma^{11}\mu_1+\sigma^{12}\mu_2+\sigma^{13}\mu_3\right)\\
\frac{R}{4w_5}\left(2\sigma^{11}w_{11}+4\sigma^{12}w_{12}+\sigma^{13}(w_3-w_4)\right)-\frac{w_{10}}{w_5}\left(\sigma^{11}\mu_1+\sigma^{12}\mu_2+\sigma^{13}\mu_3\right)\\
\frac{R}{4w_5}\left(\sigma^{11}(w_1-w_2)+\sigma^{12}(w_3-w_4)+\sigma^{13}(w_7+w_5)\right)-\frac{w_6}{2w_5}\left(\sigma^{11}\mu_1+\sigma^{12}\mu_2+\sigma^{13}\mu_3\right)\\
\frac{R}{8w_5}\left(2\sigma^{12}w_{11}+4\sigma^{22}w_{12}+\sigma^{23}(w_3-w_4)\right)-\frac{w_{10}}{2w_5}\left(\sigma^{12}\mu_1+\sigma^{22}\mu_2+\sigma^{23}\mu_3\right)\\
\frac{R}{4w_5}\left(\sigma^{12}(w_1-w_2)+\sigma^{22}(w_3-w_4)+\sigma^{23}(w_7+w_5)\right)-\frac{w_6}{2w_5}\left(\sigma^{12}\mu_1+\sigma^{22}\mu_2+\sigma^{23}\mu_3\right)\\
\frac{R}{8w_5}\left(\sigma^{13}(w_1-w_2)+\sigma^{23}(w_3-w_4)+\sigma^{33}(w_7+w_5)\right)-\frac{w_6}{4w_5}\left(\sigma^{13}\mu_1+\sigma^{23}\mu_2+\sigma^{33}\mu_3\right)
\end{bmatrix}
$$

Since $D_{\mu,\Sigma^u}SEP(\mu,\Sigma^u) = T_3\circ T_2\circ T_1(\mu,\Sigma^u)$ we may apply the chain rule to obtain the second derivative as a matrix product:

$$D^2_{\mu,\Sigma^u}SEP(\mu,\Sigma^u) = DT_3(T_2\circ T_1(\mu,\Sigma^u))\cdot DT_2(T_1(\mu,\Sigma^u))\cdot DT_1(\mu,\Sigma^u). \qquad (7)$$

The simplicity of the functions T_1, T_2 and T_3 means that the derivatives DT_1, DT_2 and DT_3 are easily obtained and then the rather large matrix products involved in $DT_3\cdot DT_2\cdot DT_1$ can be left to the computer leading to a rather painless computation of $D^2_{\mu,\Sigma^u}SEP$.

DT_1 (the total derivative of the function T_1) is the 10x9 matrix consisting of the 9x9 identity matrix (from $\frac{d(\mu,\Sigma^u)}{d(\mu,\Sigma^u)}$), with the tenth row consisting of the partials of SEP with respect to mu and sigma.

$$DT_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\partial SEP}{\partial \mu_1} & \frac{\partial SEP}{\partial \mu_2} & \frac{\partial SEP}{\partial \mu_3} & \frac{\partial SEP}{\partial \sigma_{11}} & \frac{\partial SEP}{\partial \sigma_{12}} & \frac{\partial SEP}{\partial \sigma_{13}} & \frac{\partial SEP}{\partial \sigma_{22}} & \frac{\partial SEP}{\partial \sigma_{23}} & \frac{\partial SEP}{\partial \sigma_{33}} \end{bmatrix}.$$

DT_2 is the 22x10 matrix consisting of the 10x10 identity matrix (from $\frac{d(\mu,\Sigma^u,R)}{d(\mu,\Sigma^u,R)}$), with rows 11 to 22 consisting of the 12x10 sub-matrix of partials of W with respect to μ, Σ^u and R.

$$DT_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\partial w_1}{\partial \mu_1} & \frac{\partial w_1}{\partial \mu_2} & \frac{\partial w_1}{\partial \mu_3} & \frac{\partial w_1}{\partial \sigma_{11}} & \frac{\partial w_1}{\partial \sigma_{12}} & \frac{\partial w_1}{\partial \sigma_{13}} & \frac{\partial w_1}{\partial \sigma_{22}} & \frac{\partial w_1}{\partial \sigma_{23}} & \frac{\partial w_1}{\partial \sigma_{33}} & \frac{\partial w_1}{\partial R} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial w_{12}}{\partial \mu_1} & \frac{\partial w_{12}}{\partial \mu_2} & \frac{\partial w_{12}}{\partial \mu_3} & \frac{\partial w_{12}}{\partial \sigma_{11}} & \frac{\partial w_{12}}{\partial \sigma_{12}} & \frac{\partial w_{12}}{\partial \sigma_{13}} & \frac{\partial w_{12}}{\partial \sigma_{22}} & \frac{\partial w_{12}}{\partial \sigma_{23}} & \frac{\partial w_{12}}{\partial \sigma_{33}} & \frac{\partial w_{12}}{\partial R} \end{bmatrix}$$

DT_3 is the 9x22 matrix consisting of the partials of $u_i (i=1,\ldots,9)$ with respect to μ, Σ^u, R and W.

$$DT_3 = \begin{bmatrix} \frac{\partial u_1}{\partial \mu}(1x3) & \frac{\partial u_1}{\partial \Sigma^u}(1x6) & \frac{\partial u_1}{\partial R}(1x1) & \frac{\partial u_1}{\partial W}(1x12) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial u_9}{\partial \mu}(1x3) & \frac{\partial u_9}{\partial \Sigma^u}(1x6) & \frac{\partial u_9}{\partial R}(1x1) & \frac{\partial u_9}{\partial W}(1x12) \end{bmatrix}$$

Thus, in order to obtain the second degree variance estimate given in equation (2) we must solve for the values of all elements of DT_1, DT_2 and DT_3. Once these values are computed, the solution is found by the matrix multiplication shown in equation (7), yielding the 9x9 matrix of second partials of SEP with respect to mu and sigma.

The values of the DT_1 elements are computed from equations (5) and (6). The determination of DT_2 requires the evaluation of $\frac{\partial W}{\partial \mu}$, $\frac{\partial W}{\partial \Sigma^u}$ and $\frac{\partial W}{\partial R}$ which is fully described in appendix B.

Finally, to solve for the elements of DT_3 it is necessary to solve for each of the partials given above. The evaluation of these partials is given in appendix C.

III. IMPLEMENTATION AND TESTING

A. PROGRAMMING

The calculation of $\sigma^2_{S\hat{E}P}$ the second order approximation for variance of SEP (when μ and Σ are known) has been implemented in the FORTRAN program SEPCMP which is provided in appendix D. The following steps are followed to obtain the approximation:

- Input the distribution mean (μ) and variance-covariance matrix elements (Σ^u).
- Compute SEP using μ and Σ.
- Compute the values of each element of the matrices DT1, DT2 and DT3 and then form the matrix product comprising the second derivative of SEP with respect to μ and Σ^u. The first derivative of SEP with respect to μ and Σ^u is then given by row 10 of the matrix DT1.
- Calculate the variance-covariance matrices of $\hat{\mu}$ (P_μ) and of $\hat{\Sigma}$ (P_Σ) from Σ.
- Form the SEP variance approximation using equation (2).

Computation of SEP given μ and Σ^u is based on the PL-1 program RAP provided by JHU-APL. RAP is a general program which provides the radius of coverage for any probability in the interval (0,1) for either 2 or 3 dimensions. Those portions of the program applicable to this problem, ie. probability = 0.5 and 3 dimensions, have been used in the FORTRAN subroutine FINDSEP to obtain SEP. Derivation of the equation used to compute SEP is given by L. S. Simpkins [Ref. 6] and is implemented in the subroutine FINDSEP using Gaussian

Quadrature (via the IMSL subroutine QTWODQ) to evaluate the required double integral.

Gaussian Quadrature is also used to evaluate each of the double integrals in the W, CC, CS, SC and SS functions, which are required in the evaluation of the matrices DT1, DT2 and DT3.

B. TESTING

The purpose of deriving the second order approximation of the variance of $\hat{SEP}$ is to yield more accurate confidence intervals for SEP. The approximation of these confidence intervals is derived from the fact that since $\hat{\mu}$ and $\hat{\Sigma}$ are maximum likelihood estimators of μ and Σ and thus $\hat{SEP}$ is asymptotically Normal with mean SEP. Thus, for example, an approximate 95% confidence interval for SEP is given by $\hat{SEP} \pm 1.96\sqrt{var(\hat{SEP})}$.

In order to test the affect of using the second order approximation for the variance of $\hat{SEP}$, tests of confidence interval coverage were conducted as follows:

- Sample 30 random vectors from $N(\mu,\Sigma)$ and compute $\hat{\mu}$ and $\hat{\Sigma}$ for the sample.
- Calculate $\hat{SEP}$ from $\hat{\mu}$ and $\hat{\Sigma}$.
- Calculate $\hat{\sigma}^2_{\hat{SEP}}$ by replacing $\hat{\mu}$ and $\hat{\Sigma}$ for μ and Σ in both the first order and second order approximations for the variance of $\hat{SEP}$.
- Compute the large sample approximation 95% confidence intervals for SEP using $\hat{SEP} \pm 1.96\sqrt{\hat{\sigma}^2_{\hat{SEP}}}$.

- Repeat 1000 times to calculate the percentage of times that confidence intervals based on each approximation cover the true SEP.
- For each input variance-covariance matrix, Σ, repeat the test at different magnitudes of the mean.

Trivariate normal random vectors are obtained using the IMSL subroutines DCHFAC and DRNMVN. DCHFAC performs Cholesky factorization of the distribution SIGMA which is then used as an input to DRNMVN which returns the desired random vector.

The estimates of P_μ and P_Σ are found by replacing μ and Σ with $\hat{\mu}$ and $\hat{\Sigma}$ in the formulations of P_μ and P_Σ given in Chapter I. The results of this test are summarized in Table 1.

TABLE 1. 95 PCT CONFIDENCE INTERVAL COVERAGE

SIGMA	MU	FIRST ORDER CI	SECOND ORDER CI
1.0 0.8 0.9 0.8 1.0 0.8 0.9 0.8 1.0	0 0 0	94.6	95.2
	1.0 1.2 0.8	92.8	93.3
100 -160 270 -160 400 -480 270 -480 900	0 0 0	93.3	94.2
	100 250 500	95.6	95.8
10,000 3,000 2,000 3,000 10,000 3,000 2,000 3,000 10,000	0 0 0	93.3	93.8
	8,000 10,000 12,000	93.5	94.0

These results indicate that there is little accuracy to be gained by using the second order variance approximation. There is, however, a situation in which the second order approximation becomes important. This case has been described by K. V. Kitzman [Ref. 3] and is paraphrased here.

Weapons system targeting can be measured from separate subsystems (such as navigation, guidance, postboost, etc.). Suppose that there are n independent subsystems and let $X_{i1},\ldots,X_{im} \overset{iid}{\sim} N(\mu,\Sigma)$ represent the errors for the i^{th} subsystem

i = 1,...,n, then the impact errors for the weapons system are

$Y_j = \sum_{i=1}^{n} X_{ij}$ j=1,...,m and $Y_1,\ldots,Y_m \overset{iid}{\sim} N(\mu_Y = n\mu, \Sigma_Y = n\Sigma)$.

If the Y's are measured directly and used to estimate μ_Y

and Σ_Y (to get the system SEP) then $P_{\mu_Y} = \frac{\Sigma_Y}{m} = \frac{n}{m}\Sigma$ and

$P_{\Sigma_Y} = \frac{2}{m}(\Sigma_Y \otimes \Sigma_Y) = \frac{2n^2}{m}(\Sigma \otimes \Sigma)$. If on the other hand the individual

X's are available from the subsystems then $\hat{\mu}_Y = \sum_{i=1}^{n} \overline{X_i}$ where

$\overline{X_i} = \frac{1}{n}\sum_{j=1}^{n} X_{ij}$ and $\hat{\Sigma}_Y = \frac{1}{m}\sum (X_{ij}-\hat{\mu}_Y)^T(X_{ij}-\hat{\mu}_Y)$. Then $P_{\mu_Y} = \frac{n}{m}\Sigma$ and

$P_{\Sigma_Y} = \frac{2n}{m}(\Sigma\otimes\Sigma)$. The significant factor is that the estimate of P_{Σ_Y} differs from the first estimate by a factor of n while the estimate of P_{μ_Y} remains unchanged. Thus as the number of subsystems becomes larger and the number of observations becomes smaller the size of P_{Σ} can become very small in relation to P_{μ}. As the mean approaches 0 it follows that $\frac{\partial SEP}{\partial \mu}$ approaches 0 as well and thus the contribution of P_{μ} to the estimate can become insignificant even though it is large in relation to the P_{Σ} term.

This affect has been examined as follows:

- Divide μ and Σ by n, the number of iid subsystems being simulated.
- Draw m random samples from the distribution given by this new μ and Σ for each of the n subsystems.
- Calculate $\hat{\mu}$ and $\hat{\Sigma}$ for each of the n subsystems.
- Sum the n estimates to get a total $\hat{\mu}$ and $\hat{\Sigma}$, the estimate for mean and variance in Y.
- Compute confidence intervals for $S\hat{E}P$ using both the first and second order approximations with $P_{\mu_Y} = \frac{n}{m}\Sigma$ and $P_{\Sigma_Y} = \frac{2n}{m}(\Sigma\otimes\Sigma)$.

This test was conducted with n = 4 and m = 5. The results of this test are given in Table 2. As can be seen, the effect is greatest when the mean is 0, and decreases as the mean becomes larger, which is what would be expected.

TABLE 2. 95 PCT CI COVERAGE, MULTIPLE SUBSYSTEMS

SIGMA	MU	FIRST ORDER CI	SECOND ORDER CI
100 -160 270 -160 400 -480 270 -480 900	0 0 0	92.4	97.6
	5 10 20	91.4	94.8
	25 50 80	89.4	92.2
	50 150 200	92.6	94.4

To test the sensitivity of the second order variance approximation procedure to the assumption that the detonations are normally distributed, tests were conducted in which random vectors were drawn from a mixed normal distribution. The test is conducted in the following manner:

- Prior to each sample obtain random variable p from $U(0,1)$ distribution.
- If p is less than .98 then draw a sample from $N(\mu,\Sigma)$ and proceed.
- if p is greater than .98 then draw a sample from $N(\mu,10*\Sigma)$ and proceed.
- Continue until 30 samples have been drawn, then compute all values and proceed as above.

This procedure simulates a distribution with the same mean but larger tails than the normal distribution. The confidence interval coverages are then compared to those obtained in the base case from the same input distribution. The results of this test are summarized in Table 3.

TABLE 3. 95 PCT CI COVERAGE, MIXED DISTRIBUTION

SIGMA	MU	FIRST ORDER CI	SECOND ORDER CI
1.0 0.8 0.9 0.8 1.0 0.8 0.9 0.8 1.0	0 0 0	86.6	87.6
	1.0 1.2 0.8	91.2	91.8
100 -160 270 -160 400 -480 270 -480 900	0 0 0	88.2	88.4
	100 250 500	91.5	92.4
10,000 3,000 2,000 3,000 10,000 3,000 2,000 3,000 10,000	0 0 0	79.3	80.4
	8,000 10,000 12,000	85.6	86.2

IV. CONCLUSIONS

Examination of the results in Table 1 shows that the second order approximation procedure provides a slight improvement in the 95% confidence interval coverage. The real utility of this method is brought out by the results of the multiple subsystem test shown in Table 2, where confidence interval coverage improves by as much as 5%. Since this case represents the actual situation in missile testing (i.e. very small mean and independent system observations) it is reasonable to state that use of this procedure will improve the estimate of the variance in SEP. Furthermore it seems highly unlikely that there would be any advantage to be gained by pursuing a higher order estimate of the variance of SEP.

The mixed distribution test shows that the second order approximation is only slightly more robust than the first order approximation if the distribution is actually from a mixed and not a true normal. It also indicates that if the variance is extremely large then the affect of the mixing is greater than for the smaller values and that only a few extreme values are required to greatly reduce the confidence interval for the estimate.

APPENDIX A

In order to derive the equation for the second order approximation for variance of SEP (equation (1)) we start by noting that the second order Taylor Series Expression for the SEP function is given by (ignoring the remainder term):

$$SEP(\hat{\lambda}) = SEP(\lambda)+[D_{\mu,\Sigma}SEP(\lambda)]^T\Delta\lambda+\frac{1}{2}\Delta\lambda^T[D^2_{\mu,\Sigma}SEP(\lambda)]\Delta\lambda$$

where λ, $D_{\mu,\Sigma}SEP$, and $D^2_{\mu,\Sigma}SEP$ are defined in the text, $\hat{\lambda}$ is the vector consisting of the maximum likelihood estimators for each element of λ and $\Delta\lambda = \hat{\lambda}-\lambda$. Our goal then is to find an approximation for the variance of SEP($\hat{\lambda}$). Because SEP($\hat{\lambda}$) is asymptotically unbiased and the mean square error

$$E[(SEP(\hat{\lambda})-SEP(\lambda))^2] = Var(SEP(\hat{\lambda}))+(E[SEP(\hat{\lambda})]-SEP(\lambda))^2,$$

we can get this approximation by looking at $E[SEP(\hat{\lambda})-SEP(\lambda)]^2$.

Applying the tensor identity $\underline{ABC} = A \otimes C^T \underline{B}$ to the last term and then transposing we can write the second order Taylor equation as:

$$
\begin{aligned}
SEP(\hat{\lambda}) - SEP(\lambda) &\approx [D_{\mu,\Sigma} SEP(\lambda)]^T \Delta\lambda + \frac{1}{2} (\Delta\lambda^T \otimes \Delta\lambda^T) \left[\underline{D^2_{\mu,\Sigma} SEP(\lambda)} \right] \\
&= [D_{\mu,\Sigma} SEP(\lambda)]^T \Delta\lambda + \frac{1}{2} \left[\underline{D^2_{\mu,\Sigma} SEP(\lambda)} \right]^T (\Delta\lambda \otimes \Delta\lambda)
\end{aligned}
$$

Now we note that since $E(\hat{\lambda}) = \lambda$ it follows that $E(\Delta\lambda) = 0$. If we square each side and take the expected value we see that:

$$
\begin{aligned}
E[SEP(\hat{\lambda}) - SEP(\lambda)]^2 \approx\ & [D_{\mu,\Sigma^u} SEP(\lambda)]^T E[\Delta\lambda \Delta\lambda^T] [D_{\mu,\Sigma^u} SEP(\lambda)] \\
& + \frac{1}{4} \left[\underline{D^2_{\mu,\Sigma^u} SEP(\lambda)} \right]^T E[(\Delta\lambda \otimes \Delta\lambda)(\Delta\lambda \otimes \Delta\lambda)^T] \left[\underline{D^2_{\mu,\Sigma^u} SEP(\lambda)} \right]
\end{aligned}
$$

In order to evaluate $E[(\Delta\lambda \otimes \Delta\lambda)(\Delta\lambda \otimes \Delta\lambda)^T]$ we note that $var(\Delta\lambda \otimes \Delta\lambda) = 2(SR)(P \otimes P)(SR)^T$ [Ref. 7:p. 44] where the matrix SR has the following qualities [Ref. 7: pp. 32-38]:

- SR is symmetric
- (SR)(SR) = SR
- $2(SR)(\underline{u} \otimes \underline{v}) = \underline{u} \otimes \underline{v} + \underline{v} \otimes \underline{u}$

for example, if u and v are of dimension 2 then

$$SR = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then since,

$$E[(\Delta\lambda\otimes\Delta\lambda)(\Delta\lambda\otimes\Delta\lambda)^T] = var(\Delta\lambda\otimes\Delta\lambda)+E(\Delta\lambda\otimes\Delta\lambda)[E(\Delta\lambda\otimes\Delta\lambda)]^T$$

and

$$\Delta\lambda\Delta\lambda^T = \Delta\lambda\otimes\Delta\lambda$$

if we define P to be $E[\Delta\lambda\Delta\lambda^T] = var(\Delta\lambda)$ we see that

$$\begin{aligned} var(\Delta\lambda\otimes\Delta\lambda)+E(\Delta\lambda\otimes\Delta\lambda)[E(\Delta\lambda\otimes\Delta\lambda)]^T &= 2SR(P\otimes P)(SR)^T+\underline{P}\,\underline{P}^T \\ &= 2(SR)^T(P\otimes P)(SR)+\underline{P}\,\underline{P}^T. \end{aligned}$$

Since $D^2_{\mu,\Sigma}SEP$ is a symmetric matrix it follows that $(SR)\underline{D^2_{\mu,\Sigma}SEP} = \underline{D^2_{\mu,\Sigma}SEP}$ and we see that:

$$
\begin{aligned}
E[SEP(\hat{\lambda})-SEP(\lambda)]^2 \approx\ & [D_{\mu,\Sigma}SEP(\lambda)]^T P[D_{\mu,\Sigma}SEP(\lambda)] \\
& + \frac{1}{2}\left[\underline{D^2_{\mu,\Sigma}SEP(\lambda)}\right]^T P\otimes P\left[\underline{D^2_{\mu,\Sigma}SEP(\lambda)}\right] \\
& + \frac{1}{4}\left[\underline{D^2_{\mu,\Sigma}SEP(\lambda)}\right]^T \underline{P}\,\underline{P}^T\left[\underline{D^2_{\mu,\Sigma}SEP(\lambda)}\right],
\end{aligned}
$$

which in turn is approximately equal to the variance of SEP($\hat{\lambda}$).

Finally, since Σ is a 3x3 symmetric matrix, we need only describe the unique terms (which we have denoted by Σ^u) to arrive at the second order equation for SEP given by equation (1).

APPENDIX B

A. APPROACH

It is seen that each of the 12 W_i functions consists of a double integral of cos and sin functions multiplied by the function $g(R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta)$. Each of the required derivatives carries through the integrals, so that we need to evaluate each in respect to the function g.

We will first evaluate each of the partials of g, and then carry these through to the functions W_i.

Recall,

$$
\begin{aligned}
g(R,\theta,\phi) = \exp\Big\{-\frac{1}{2}\big[&(R\sin\theta\cos\phi-\mu_1)^2\sigma^{11} \\
&+ 2(R\sin\theta\cos\phi-\mu_1)(R\sin\theta\sin\phi-\mu_2)\sigma^{12} \\
&+ 2(R\sin\theta\cos\phi-\mu_1)(R\cos\theta-\mu_3)\sigma^{13} \\
&+ (R\sin\theta\sin\phi-\mu_2)^2\sigma^{22} \\
&+ 2(R\sin\theta\sin\phi-\mu_2)(R\cos\theta-\mu_3)\sigma^{23} \\
&+ (R\cos\theta-\mu_3)^2\sigma^{33}\big]\Big\}.
\end{aligned}
$$

If we define the vectors A_1, A_2, A_3 and A_4 by:

$$A_1 = \begin{bmatrix} \sin^2\theta\cos^2\phi \\ 2\sin^2\theta\cos\phi\sin\phi \\ 2\sin\theta\cos\theta\cos\phi \\ \sin^2\theta\sin^2\phi \\ 2\sin\theta\cos\theta\sin\phi \\ \cos^2\theta \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \sin\theta\cos\phi\mu_1 \\ \sin\theta\sin\phi\mu_1+\sin\theta\cos\phi\mu_2 \\ \cos\theta\mu_1+\sin\theta\cos\phi\mu_3 \\ \sin\theta\sin\phi\mu_2 \\ \cos\theta\mu_2+\sin\theta\sin\phi\mu_3 \\ \cos\theta\mu_3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix}\mu_1^2 & 2\mu_1\mu_2 & 2\mu_1\mu_3 & \mu_2^2 & 2\mu_2\mu_3 & \mu_3^2\end{bmatrix}^T,$$

$$A_4 = \begin{bmatrix} R\sin\theta\cos\phi-\mu_1 \\ R\sin\theta\sin\phi-\mu_2 \\ R\cos\theta-\mu_3 \end{bmatrix}.$$

Then treating R as an independent parameter and taking the partial with respect to R we have:

$$
\begin{aligned}
\frac{\partial g}{\partial R} = -g\cdot\{&\sigma^{11}(R\sin^2\theta\cos^2\phi-\mu_1\sin\theta\cos\phi) \\
&+\sigma^{12}(2R\sin^2\theta\cos\phi\sin\phi-\mu_1\sin\theta\sin\phi-\mu_2\sin\theta\cos\phi) \\
&+\sigma^{13}(2R\sin\theta\cos\theta\cos\phi-\mu_1\cos\theta-\mu_3\sin\theta\cos\phi) \\
&+\sigma^{22}(R\sin^2\theta\sin^2\phi-\mu_2\sin\theta\sin\phi) \\
&+\sigma^{23}(2R\sin\theta\sin\phi\cos\theta-\mu_2\cos\theta-\mu_3\sin\theta\sin\phi) \\
&+\sigma^{33}(R\cos^2\theta-\mu_3\cos\theta)\}
\end{aligned}
$$

$$
= -(\Sigma^{-1})^{u}[R\cdot A_1-A_2].
$$

Similarly, we derive:

$$
\begin{aligned}
\frac{\partial g}{\partial \mu_1} &= g\cdot\{\sigma^{11}(R\sin\theta\cos\phi-\mu_1)+\sigma^{12}(R\sin\theta\sin\phi-\mu_2)+\sigma^{13}(R\cos\theta-\mu_3)\} \\
&= g\cdot\{[\sigma^{11}\ \ \sigma^{12}\ \ \sigma^{13}]A_4\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial g}{\partial \mu_2} &= g\cdot\{\sigma^{12}(R\sin\theta\cos\phi-\mu_1)+\sigma^{22}(R\sin\theta\sin\phi-\mu_2)+\sigma^{23}(R\cos\theta-\mu_3)\} \\
&= g\cdot\{[\sigma^{12}\ \ \sigma^{22}\ \ \sigma^{23}]A_4\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial g}{\partial \mu_3} &= g\cdot\{\sigma^{13}(R\sin\theta\cos\phi-\mu_1)+\sigma^{23}(R\sin\theta\sin\phi-\mu_2)+\sigma^{33}(R\cos\theta-\mu_3)\}. \\
&= g\cdot\{[\sigma^{13}\ \ \sigma^{23}\ \ \sigma^{33}]A_4\}
\end{aligned}
$$

In order to derive the partials with respect to sigma we will first generate the partials $\frac{\partial\sigma^{ij}}{\partial\sigma_{kl}}$ and the partials of $\frac{1}{q}$ with respect to σ_{kl}. These values are given below.

	$\underline{\sigma^{11}}$	$\underline{\sigma^{12}}$	$\underline{\sigma^{13}}$	$\underline{\sigma^{22}}$
$\frac{\partial}{\partial\sigma_{11}}$	$-\sigma^{11}\sigma^{11}$	$-\sigma^{12}\sigma^{11}$	$-\sigma^{13}\sigma^{11}$	$\frac{\sigma_{33}}{q^2}-\sigma^{22}\sigma^{11}$
$\frac{\partial}{\partial\sigma_{12}}$	$-2\sigma^{11}\sigma^{12}$	$\frac{-\sigma_{33}}{q^2}-2\sigma^{12}\sigma^{12}$	$\frac{\sigma_{23}}{q^2}-2\sigma^{13}\sigma^{12}$	$-2\sigma^{22}\sigma^{12}$
$\frac{\partial}{\partial\sigma_{13}}$	$-2\sigma^{11}\sigma^{13}$	$\frac{\sigma_{23}}{q^2}-2\sigma^{12}\sigma^{13}$	$\frac{-\sigma_{22}}{q^2}-2\sigma^{13}\sigma^{13}$	$\frac{-2\sigma_{13}}{q^2}-2\sigma^{22}\sigma^{13}$
$\frac{\partial}{\partial\sigma_{22}}$	$\frac{\sigma_{33}}{q^2}-\sigma^{11}\sigma^{22}$	$-\sigma^{12}\sigma^{22}$	$\frac{-\sigma_{13}}{q^2}-\sigma^{13}\sigma^{22}$	$-\sigma^{22}\sigma^{22}$
$\frac{\partial}{\partial\sigma_{23}}$	$\frac{-2\sigma_{23}}{q^2}-2\sigma^{11}\sigma^{23}$	$\frac{\sigma_{13}}{q^2}-2\sigma^{12}\sigma^{23}$	$\frac{\sigma_{12}}{q^2}-2\sigma^{13}\sigma^{23}$	$-2\sigma^{22}\sigma^{23}$
$\frac{\partial}{\partial\sigma_{33}}$	$\frac{\sigma_{22}}{q^2}-\sigma^{11}\sigma^{33}$	$\frac{-\sigma_{12}}{q^2}-\sigma^{12}\sigma^{33}$	$-\sigma^{13}\sigma^{33}$	$\frac{\sigma_{11}}{q^2}-\sigma^{22}\sigma^{33}$

	$\underline{\sigma^{23}}$	$\underline{\sigma^{33}}$	$\underline{q^{-1}}$
$\frac{\partial}{\partial\sigma_{11}}$	$\frac{-\sigma_{23}}{q^2}-\sigma^{23}\sigma^{11}$	$\frac{\sigma_{22}}{q^2}-\sigma^{33}\sigma^{11}$	$\frac{-\sigma^{11}}{2q}$
$\frac{\partial}{\partial\sigma_{12}}$	$\frac{\sigma_{13}}{q^2}-2\sigma^{23}\sigma^{12}$	$\frac{-2\sigma_{12}}{q^2}-2\sigma^{33}\sigma^{12}$	$\frac{-\sigma^{12}}{q}$
$\frac{\partial}{\partial\sigma_{13}}$	$\frac{\sigma_{12}}{q^2}-2\sigma^{23}\sigma^{13}$	$-2\sigma^{33}\sigma^{13}$	$\frac{-\sigma^{13}}{q}$
$\frac{\partial}{\partial\sigma_{22}}$	$-\sigma^{23}\sigma^{22}$	$\frac{\sigma_{11}}{q^2}-\sigma^{33}\sigma^{22}$	$\frac{-\sigma^{22}}{2q}$
$\frac{\partial}{\partial\sigma_{23}}$	$\frac{-\sigma_{11}}{q^2}-2\sigma^{23}\sigma^{23}$	$-2\sigma^{33}\sigma^{23}$	$\frac{-\sigma^{23}}{q}$
$\frac{\partial}{\partial\sigma_{33}}$	$-\sigma^{23}\sigma^{33}$	$-\sigma^{33}\sigma^{33}$	$\frac{-\sigma^{33}}{2q}$

We can now use these values to find the desired partials:

$$\frac{\partial g}{\partial \sigma_{11}} = g\cdot\left\{\frac{1}{2}\left[(R\sin\theta\cos\phi-\mu_1)^2(\sigma^{11})^2\right.\right.$$

$$+2(R\sin\theta\cos\phi-\mu_1)(R\sin\theta\sin\phi-\mu_2)\sigma^{12}\sigma^{11}$$

$$+2(R\sin\theta\cos\phi-\mu_1)(R\cos\theta-\mu_3)\sigma^{13}\sigma^{11}$$

$$+(R\sin\theta\sin\phi-\mu_2)^2\left(\sigma^{22}\sigma^{11}-\frac{\sigma_{33}}{q^2}\right)$$

$$+2(R\sin\theta\sin\phi-\mu_2)(R\cos\theta-\mu_3)\left(\sigma^{23}\sigma^{11}+\frac{\sigma_{23}}{q^2}\right)$$

$$\left.\left.+(R\cos\theta-\mu_3)^2\left(\sigma^{33}\sigma^{11}-\frac{\sigma_{22}}{q^2}\right)\right]-\frac{\sigma^{11}}{2}\right\}$$

$$= g\cdot\left\{\frac{1}{2}\left[\sigma^{11}R^2(\Sigma^{-1})^u A_1-2R\sigma^{11}(\Sigma^{-1})^u A_2+\sigma^{11}(\Sigma^{-1})^u A_3\right]\right.$$

$$\left.+\begin{bmatrix}0 & 0 & 0 & \frac{-\sigma_{33}}{q^2} & \frac{\sigma_{23}}{q^2} & \frac{-\sigma_{22}}{q^2}\end{bmatrix}\left[R^2A_1-2RA_2+A_3\right]\right\}$$

$$= g\cdot\left\{\frac{1}{2}\left[\sigma^{11}(\Sigma^{-1})^u+\begin{pmatrix}0 & 0 & 0 & \frac{-\sigma_{33}}{q^2} & \frac{\sigma_{23}}{q^2} & \frac{-\sigma_{22}}{q^2}\end{pmatrix}\right]\left[R^2A_1-2RA_2+A_3\right]-\frac{\sigma^{11}}{2}\right\}$$

Following this same methodology it is easy to see that:

$$\frac{\partial g}{\partial \sigma_{12}} = g\cdot\left\{\left[\sigma^{12}(\Sigma^{-1})^u+\begin{pmatrix}0 & \frac{\sigma_{33}}{2q^2} & \frac{-\sigma_{23}}{2q^2} & 0 & \frac{-\sigma_{13}}{2q^2} & \frac{\sigma_{12}}{q^2}\end{pmatrix}\right]\left[R^2A_1-2RA_2+A_3\right]-\sigma^{12}\right\}$$

$$\frac{\partial g}{\partial \sigma_{13}} = g\cdot\left\{\left[\sigma^{13}(\Sigma^{-1})^u+\begin{pmatrix}0 & \frac{-\sigma_{23}}{2q^2} & \frac{\sigma_{22}}{2q^2} & \frac{\sigma_{13}}{q^2} & \frac{-\sigma_{12}}{2q^2} & 0\end{pmatrix}\right]\left[R^2A_1-2RA_2+A_3\right]-\sigma^{13}\right\}$$

$$\frac{\partial g}{\partial \sigma_{22}} = g\cdot\left\{\frac{1}{2}\left[\sigma^{22}(\Sigma^{-1})^u+\begin{pmatrix}\frac{-\sigma_{33}}{q^2} & 0 & \frac{\sigma_{13}}{q^2} & 0 & 0 & \frac{-\sigma_{11}}{q^2}\end{pmatrix}\right]\left[R^2A_1-2RA_2+A_3\right]-\frac{\sigma^{22}}{2}\right\}$$

$$\frac{\partial g}{\partial \sigma_{23}} = g\cdot\left\{\sigma^{23}(\Sigma^{-1})^{u}+\left(\frac{\sigma_{23}}{q^2}\ \ \frac{-\sigma_{13}}{2q^2}\ \ \frac{-\sigma_{12}}{2q^2}\ \ 0\ \ \frac{\sigma_{11}}{2q^2}\ \ 0\right)\right]\left[R^2A_1-2RA_2+A_3\right]-\sigma^{23}\right\}$$

$$\frac{\partial g}{\partial \sigma_{33}} = g\cdot\left\{\frac{1}{2}\left[\sigma^{33}(\Sigma^{-1})^{u}+\left(\frac{-\sigma_{22}}{q^2}\ \ \frac{\sigma_{12}}{q^2}\ \ 0\ \ \frac{-\sigma_{11}}{q^2}\ \ 0\ \ 0\right)\right]\left[R^2A_1-2RA_2+A_3\right]-\frac{\sigma^{33}}{2}\right\}$$

We are now able to calculate each of the partials of the W_i by simply computing the effects of the four A vectors on the original W functions and applying the formulas derived on the previous pages. We will carry through the complete derivation for W_1 and then show the solutions for the other cases. Since the vector A_3 does not involve trigonometric functions, it will not be necessary to compute any partials for it.

B. PARTIALS OF W1

$W_1 = CC_{1,1,1,1} = \int_0^{2\pi}\int_0^{\pi}\cos\theta\cos\phi\, g(R,\theta,\phi)\,d\theta\, d\phi$ so we can see that

the integrals of gA will yield:

$$\int_0^{2\pi}\int_0^{\pi} \cos\theta\cos\phi\, g\cdot A_1 = \int_0^{2\pi}\int_0^{\pi} g\cdot\begin{bmatrix} \sin^2\theta\cos\theta\cos^3\phi \\ 2\sin^2\theta\cos\theta\cos^2\phi\sin\phi \\ 2\sin\theta\cos^2\theta\cos^2\phi \\ \sin^2\theta\cos\theta\cos\phi\sin^2\phi \\ 2\sin\theta\cos^2\theta\sin\phi\cos\phi \\ \cos^3\theta\cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} CC_{1,1,3,1}-CC_{3,1,3,1} \\ 2(CS_{1,1,1,1}-CS_{3,1,1,1}-CS_{1,1,3,1}+CS_{3,1,3,1}) \\ 2(SC_{1,1,2,1}-SC_{3,1,2,1}) \\ CC_{1,1,1,1}-CC_{3,1,1,1}-CC_{1,1,3,1}+CC_{3,1,3,1} \\ SS_{1,1,1,2}-SS_{3,1,1,2} \\ CC_{3,1,1,1} \end{bmatrix}$$

$$\int_0^{2\pi}\int_0^{\pi}\cos\theta\cos\phi\, g\cdot A_2 = \int_0^{2\pi}\int_0^{\pi} g\cdot\begin{bmatrix} \sin\theta\cos\theta\cos^2\phi\mu_1 \\ \sin\theta\cos\theta\cos\phi\sin\phi\mu_1+\sin\theta\cos\theta\cos^2\phi\mu_2 \\ \cos^2\theta\cos\phi\mu_1+\sin\theta\cos\theta\cos^2\phi\mu_3 \\ \sin\theta\cos\theta\cos\phi\sin\phi\mu_2 \\ \cos^2\theta\cos\phi\mu_2+\sin\theta\cos\theta\cos\phi\sin\phi\mu_3 \\ \cos^2\theta\cos\phi\mu_3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2\mu_1 SC_{1,2,2,1} \\ \mu_1 SS_{1,2,1,2}+2\mu_2 SC_{1,2,2,1} \\ 4\mu_1 CC_{2,1,1,1}+2\mu_3 SC_{1,2,2,1} \\ \mu_2 SS_{1,2,1,2} \\ 4\mu_2 CC_{2,1,1,1}+\mu_3 SS_{1,2,1,2} \\ 4\mu_3 CC_{2,1,1,1} \end{bmatrix}$$

$$\int_0^{2\pi}\int_0^{\pi}\cos\theta\cos\phi g\cdot A_4 = \int_0^{2\pi}\int_0^{\pi} g\begin{bmatrix} R\sin\theta\cos\theta\cos^2\phi-\cos\theta\cos\phi\mu_1 \\ R\sin\theta\cos\theta\cos\phi\sin\phi-\cos\theta\cos\phi\mu_2 \\ R\cos^2\theta\cos\phi-\cos\theta\cos\phi\mu_3 \end{bmatrix}.$$

$$= \begin{bmatrix} \frac{R}{2}\sin2\theta\cos^2\phi-\mu_1\cos\theta\cos\phi \\ \frac{R}{4}\sin2\theta\sin2\phi-\mu_2\cos\theta\cos\phi \\ R\cos^2\theta\cos\phi-\mu_3\cos\theta\cos\phi \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2SC_{1,2,2,1}R-4\mu_1 CC_{1,1,1,1} \\ SS_{1,2,1,2}R-4\mu_2 CC_{1,1,1,1} \\ 4CC_{2,1,1,1}R-4\mu_3 CC_{1,1,1,1} \end{bmatrix}$$

For future use, we point out that the partials of A_2 and A_4 involve only 3 distinct trigonometric terms, which we will define by:

$$B = \begin{cases} B_1 = \sin\theta\cos\phi \\ B_2 = \sin\theta\sin\phi \\ B_3 = \cos\theta \end{cases}$$

then we see that

$$A_2 = \begin{bmatrix} B_1\mu_1 \\ B_2\mu_1+B_1\mu_2 \\ B_3\mu_1+B_1\mu_3 \\ B_2\mu_2 \\ B_3\mu_2+B_2\mu_3 \\ B_3\mu_3 \end{bmatrix}$$

and

$$A_4 = \begin{bmatrix} B_1 R - \mu_1 \\ B_2 R - \mu_2 \\ B_3 R - \mu_3 \end{bmatrix}.$$

Thus, we need only compute the effects of A_1 and B on each of the 12 W functions to obtain the desired partials.

C. PARTIALS OF W2

$W_2 = CC_{1,3,1,1} = \int_0^{2\pi}\int_0^{\pi} \cos 3\theta \cos\phi\, g(R,\theta,\phi)\, d\theta\, d\phi$ so the partials are given by:

$$\cos 3\theta \cos\phi A_1 = \begin{bmatrix} \sin^2\theta\cos 3\theta\cos^3\phi \\ 2\sin^2\theta\cos 3\theta\cos^2\phi\sin\phi \\ 2\sin\theta\cos\theta\cos 3\theta\cos^2\phi \\ \sin^2\theta\cos 3\theta\sin^2\phi\cos\phi \\ 2\sin\theta\cos\theta\cos 3\theta\sin\phi\cos\phi \\ \cos^2\theta\cos 3\theta\cos\phi \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2CC_{1,3,3,1} - CC_{1,5,3,1} - CC_{1,1,3,1} \\ 2(2CS_{1,3,1,1} - CS_{1,5,1,1} - CS_{1,1,1,1} - 2CS_{1,3,3,1} + CS_{1,5,3,1} + CS_{1,1,3,1}) \\ 2(SC_{1,5,2,1} - SC_{1,1,2,1}) \\ 2CC_{1,3,1,1} - CC_{1,5,1,1} - CC_{1,1,1,1} - 2CC_{1,3,3,1} + CC_{1,5,3,1} + CC_{1,1,3,1} \\ SS_{1,5,1,2} - SS_{1,1,1,2} \\ CC_{1,5,1,1} + 2CC_{1,3,1,1} + CC_{1,1,1,1} \end{bmatrix}$$

$$\cos 3\theta\cos\phi B = \begin{bmatrix} \sin\theta\cos 3\theta\cos^2\phi \\ \sin\theta\cos 3\theta\cos\phi\sin\phi \\ \cos\theta\cos 3\theta\cos\phi \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2(SC_{1,4,2,1}-SC_{1,2,2,1}) \\ SS_{1,4,1,2}-SS_{1,2,1,2} \\ 2(CC_{1,4,1,1}+CC_{1,2,1,1}) \end{bmatrix}$$

D. PARTIALS OF W3

$W_3 = CS_{1,1,1,1} = \int_0^{2\pi}\int_0^{\pi}\cos\theta\sin\phi g(R,\theta,\phi)d\theta d\phi$ so the partials are given by:

$$\cos\theta\sin\phi A_1 = \begin{bmatrix} \sin^2\theta\cos\theta\cos^2\phi\sin\phi \\ 2\sin^2\theta\cos\theta\cos\phi\sin^2\phi \\ 2\sin\theta\cos^2\theta\cos\phi\sin\phi \\ \sin^2\theta\cos\theta\sin^3\phi \\ 2\sin\theta\cos^2\theta\sin^2\phi \\ \cos^3\theta\sin\phi \end{bmatrix}$$

$$= \begin{bmatrix} CS_{1,1,1,1}-CS_{3,1,1,1}-CS_{1,1,3,1}+CS_{3,1,3,1} \\ 2(CC_{1,1,1,1}-CC_{1,1,3,1}-CC_{3,1,1,1}+CC_{3,1,3,1}) \\ SS_{1,1,1,2}-SS_{3,1,1,2} \\ CS_{1,1,3,1}-CS_{3,1,3,1} \\ 2(SS_{1,1,2,1}-SS_{3,1,2,1}) \\ CS_{3,1,1,1} \end{bmatrix}$$

$$\cos\theta\sin\phi B = \begin{bmatrix} \sin\theta\cos\theta\cos\phi\sin\phi \\ \sin\theta\cos\theta\sin^2\phi \\ \cos^2\theta\sin\phi \end{bmatrix} = \frac{1}{4}\begin{bmatrix} SS_{1,2,1,2} \\ 2SS_{1,2,2,1} \\ 4CS_{2,1,1,1} \end{bmatrix}$$

E. PARTIALS OF W4

$$W_4 = CS_{1,3,1,1} = \int_0^{2\pi}\int_0^{\pi} \cos 3\theta \sin\phi\, g(R,\theta,\phi)\, d\theta\, d\phi$$ so the partials

are given by:

$$\cos 3\theta \sin\phi A_1 = \begin{bmatrix} \sin^2\theta\cos 3\theta\cos^2\phi\sin\phi \\ 2\sin^2\theta\cos 3\theta\cos\phi\sin^2\phi \\ 2\sin\theta\cos\theta\cos 3\theta\cos\phi\sin\phi \\ \sin^2\theta\cos 3\theta\sin^3\phi \\ 2\sin\theta\cos\theta\cos 3\theta\sin^2\phi \\ \cos^2\theta\cos 3\theta\sin\phi \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2CS_{1,3,1,1} - 2CS_{1,3,3,1} - CS_{1,5,1,1} + CS_{1,5,3,1} - CS_{1,1,1,1} + CS_{1,1,3,1} \\ 2(2CC_{1,3,1,1} - 2CC_{1,3,3,1} - CC_{1,5,1,1} + CC_{1,5,3,1} - CC_{1,1,1,1} + CC_{1,1,3,1}) \\ SS_{1,5,1,2} - SS_{1,1,1,2} \\ 2CS_{1,3,3,1} - CS_{1,5,3,1} - CS_{1,1,3,1} \\ 2(SS_{1,5,2,1} - SS_{1,1,2,1}) \\ CS_{1,5,1,1} + 2CS_{1,3,1,1} + CS_{1,1,1,1} \end{bmatrix}$$

$$\cos 3\theta \sin\phi B = \begin{bmatrix} \sin\theta\cos 3\theta\cos\phi\sin\phi \\ \sin\theta\cos 3\theta\sin^2\phi \\ \cos\theta\cos 3\theta\sin\phi \end{bmatrix} = \frac{1}{4}\begin{bmatrix} SS_{1,4,1,2} - SS_{1,2,1,2} \\ 2(SS_{1,4,2,1} - SS_{1,2,2,1}) \\ 2(CS_{1,4,1,1} + CS_{1,2,1,1}) \end{bmatrix}$$

F. PARTIALS OF W5

$W_5 = SC_{1,1,0,0} = \int_0^{2\pi}\int_0^{\pi} \sin\theta g(R,\theta,\phi) d\theta d\phi$ so the partials are given by:

$$\sin\theta A_1 = \begin{bmatrix} \sin^3\theta\cos^2\phi \\ 2\sin^3\theta\cos\phi\sin\phi \\ 2\sin^2\theta\cos\theta\cos\phi \\ \sin^3\theta\sin^2\phi \\ 2\sin^2\theta\cos\theta\sin\phi \\ \cos^2\theta\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} SC_{3,1,2,1} \\ SS_{3,1,1,2} \\ 2(CC_{1,1,1,1} - CC_{3,1,1,1}) \\ SS_{3,1,2,1} \\ 2(CS_{1,1,1,1} - CS_{3,1,1,1}) \\ SC_{1,1,0,0} - SC_{3,1,0,0} \end{bmatrix}$$

$$\sin\theta B = \begin{bmatrix} \sin^2\theta\cos\phi \\ \sin^2\theta\sin\phi \\ \sin\theta\cos\theta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2SC_{2,1,1,1} \\ 2SS_{2,1,1,1} \\ SC_{1,2,0,0} \end{bmatrix}$$

G. PARTIALS OF W6

$W_6 = SC_{1,2,0,0} = \int_0^{2\pi}\int_0^{\pi} \sin 2\theta\, g(R,\theta,\phi)\, d\theta\, d\phi$ so the partials are given by:

$$\sin 2\theta A_1 = \begin{bmatrix} \sin^2\theta \sin 2\theta \cos^2\phi \\ 2\sin^2\theta \sin 2\theta \cos\phi \sin\phi \\ 2\sin\theta \sin 2\theta \cos\theta \cos\phi \\ \sin^2\theta \sin 2\theta \sin^2\phi \\ 2\sin\theta \sin 2\theta \cos\theta \sin\phi \\ \cos^2\theta \sin 2\theta \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2SC_{1,2,2,1} - SC_{1,4,2,1} \\ 2SS_{1,2,1,2} - SS_{1,4,1,2} \\ 2(CC_{1,2,1,1} - CC_{1,4,1,1} + 2SC_{2,1,1,1}) \\ 2SS_{1,2,2,1} - SS_{1,4,2,1} \\ 2(CS_{1,2,1,1} - CS_{1,4,1,1} + 2SS_{2,1,1,1}) \\ 2SC_{1,2,0,0} + SC_{1,4,0,0} \end{bmatrix}.$$

$$\sin 2\theta B = \begin{bmatrix} \sin\theta \sin 2\theta \cos\phi \\ \sin\theta \sin 2\theta \sin\phi \\ \sin 2\theta \cos\theta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} CC_{1,1,1,1} - CC_{1,3,1,1} \\ CS_{1,1,1,1} - CS_{1,3,1,1} \\ SC_{1,3,0,0} + SC_{1,1,0,0} \end{bmatrix}$$

H. PARTIALS OF W7

$W_7 = SC_{1,3,0,0} = \int_0^{2\pi}\int_0^{\pi}\sin 3\theta\, g(R,\theta,\phi)\,d\theta\, d\phi$ so the partials are given by:

$$\sin 3\theta A_1 = \begin{bmatrix} \sin^2\theta\sin 3\theta\cos^2\phi \\ 2\sin^2\theta\sin 3\theta\cos\phi\sin\phi \\ 2\sin\theta\sin 3\theta\cos\theta\cos\phi \\ \sin^2\theta\sin 3\theta\sin^2\phi \\ 2\sin\theta\sin 3\theta\cos\theta\sin\phi \\ \cos^2\theta\sin 3\theta \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2SC_{1,3,2,1}-SC_{1,1,2,1}-SC_{1,5,2,1} \\ 2SS_{1,3,1,2}-SS_{1,1,1,2}-SS_{1,5,1,2} \\ 2(CC_{1,1,1,1}-CC_{1,5,1,1}) \\ 2SS_{1,3,2,1}-SS_{1,1,2,1}-SS_{1,5,2,1} \\ 2(CS_{1,1,1,1}-CS_{1,5,1,1}) \\ SC_{1,5,0,0}+2SC_{1,3,0,0}+SC_{1,1,0,0} \end{bmatrix}$$

$$\sin 3\theta B = \begin{bmatrix} \sin\theta\sin 3\theta\cos\phi \\ \sin\theta\sin 3\theta\sin\phi \\ \sin 3\theta\cos\theta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} CC_{1,2,1,1}-CC_{1,4,1,1} \\ CS_{1,2,1,1}-CS_{1,4,1,1} \\ SC_{1,4,0,0}+SC_{1,2,0,0} \end{bmatrix}$$

I. PARTIALS OF W8

$W_8 = SC_{2,1,1,1} = \int_0^{2\pi}\int_0^{\pi}\sin^2\theta\cos\phi\, g(R,\theta,\phi)\,d\theta\, d\phi$ so the partials are

given by:

$$\sin^2\theta\cos\phi A_1 = \begin{bmatrix} \sin^4\theta\cos^3\phi \\ 2\sin^4\theta\cos^2\phi\sin\phi \\ 2\sin^3\theta\cos\theta\cos^2\phi \\ \sin^4\theta\sin^2\phi\cos\phi \\ 2\sin^3\theta\cos\theta\cos\phi\sin\phi \\ \cos^2\theta\sin^2\theta\cos\phi \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 8SC_{4,1,3,1} \\ 16(SS_{4,1,1,1}-SS_{4,1,3,1}) \\ 2(2SC_{1,2,2,1}-SC_{1,4,2,1}) \\ 8(SC_{4,1,1,1}-SC_{4,1,3,1}) \\ 2SS_{1,2,1,2}-SS_{1,4,1,2} \\ 8(SC_{2,1,1,1}-SC_{4,1,1,1}) \end{bmatrix}$$

$$\sin^2\theta\cos\phi B = \begin{bmatrix} \sin^3\theta\cos^2\phi \\ \sin^3\theta\sin\phi\cos\phi \\ \sin^2\theta\cos\theta\cos\phi \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2SC_{3,1,2,1} \\ SS_{3,1,1,2} \\ 2(CC_{1,1,1,1}-CC_{3,1,1,1}) \end{bmatrix}$$

J. PARTIALS OF W9

$W_9 = SC_{3,1,2,1} = \int_0^{2\pi}\int_0^{\pi} \sin^3\theta\cos^2\phi\, g(R,\theta,\phi)\, d\theta\, d\phi$ so the partials are

given by:

$$\sin^3\theta\cos^2\phi A_1 = \begin{bmatrix} \sin^5\theta\cos^4\phi \\ 2\sin^5\theta\cos^3\phi\sin\phi \\ 2\sin^4\theta\cos\theta\cos^3\phi \\ \sin^5\theta\sin^2\phi\cos^2\phi \\ 2\sin^4\theta\cos\theta\cos^2\phi\sin\phi \\ \cos^2\theta\sin^3\theta\cos^2\phi \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4SC_{5,1,4,1} \\ SS_{5,1,1,4}+2SS_{5,1,1,2} \\ 8(CC_{1,1,3,1}-2CC_{3,1,3,1}+CC_{5,1,3,1}) \\ 4(SS_{5,1,2,1}-SS_{5,1,4,1}) \\ 8(CS_{1,1,1,1}-CS_{1,1,3,1}-2CS_{3,1,1,1}+2CS_{3,1,3,1}+CS_{5,1,1,1}-CS_{5,1,3,1}) \\ 4(SC_{3,1,2,1}-SC_{5,1,2,1}) \end{bmatrix}$$

$$\sin^3\theta\cos^2\phi B = \begin{bmatrix} \sin^4\theta\cos^3\phi \\ \sin^4\theta\sin\phi\cos^2\phi \\ \sin^3\theta\cos\theta\cos^2\phi \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8SC_{4,1,3,1} \\ 8(SS_{4,1,1,1}-SS_{4,1,3,1}) \\ 2SC_{1,2,2,1}-SC_{1,4,2,1} \end{bmatrix}$$

K. PARTIALS OF W10

$W_{10} = SS_{2,1,1,1} = \int_0^{2\pi}\int_0^{\pi} \sin^2\theta \sin\phi\, g(R,\theta,\phi)\, d\theta\, d\phi$ so the partials are

given by:

$$\sin^2\theta\sin\phi A_1 = \begin{bmatrix} \sin^4\theta\cos^2\phi\sin\phi \\ 2\sin^4\theta\cos\phi\sin^2\phi \\ 2\sin^3\theta\cos\theta\cos\phi\sin\phi \\ \sin^4\theta\sin^3\phi \\ 2\sin^3\theta\cos\theta\sin^2\phi \\ \cos^2\theta\sin^2\theta\sin\phi \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 8(SS_{4,1,1,1}-SS_{4,1,3,1}) \\ 16(SC_{4,1,1,1}-SC_{4,1,3,1}) \\ 2SS_{1,2,1,2}-SS_{1,4,1,2} \\ 8SS_{4,1,3,1} \\ 4SS_{1,2,2,1}-2SS_{1,4,2,1} \\ 8(SS_{2,1,1,1}-SS_{4,1,1,1}) \end{bmatrix}$$

$$\sin^2\theta\sin\phi B = \begin{bmatrix} \sin^3\theta\cos\phi\sin\phi \\ \sin^3\theta\sin^2\phi \\ \sin^2\theta\cos\theta\sin\phi \end{bmatrix} = \frac{1}{2}\begin{bmatrix} SS_{3,1,1,2} \\ 2SS_{3,1,2,1} \\ 2(CS_{1,1,1,1}-CS_{3,1,1,1}) \end{bmatrix}$$

L. PARTIALS OF W11

$W_{11} = SS_{3,1,1,2} = \int_0^{2\pi}\int_0^{\pi} \sin^3\theta \sin 2\phi\, g(R,\theta,\phi)\, d\theta\, d\phi$ so the partials

are given by:

$$\sin^3\theta\sin 2\phi A_1 = \begin{bmatrix} \sin^5\theta\cos^2\phi\sin 2\phi \\ 2\sin^5\theta\cos\phi\sin\phi\sin 2\phi \\ 2\sin^4\theta\cos\theta\cos\phi\sin 2\phi \\ \sin^5\theta\sin^2\phi\sin 2\phi \\ 2\sin^4\theta\cos\theta\sin\phi\sin 2\phi \\ \cos^2\theta\sin^3\theta\sin 2\phi \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} SS_{5,1,1,4}+2SS_{5,1,1,2} \\ 2(2SC_{5,1,2,1}-SC_{5,1,1,4}-SC_{5,1,1,2}) \\ 4(CS_{1,1,1,3}+CS_{1,1,1,1}-2CS_{3,1,1,3}-2CS_{3,1,1,1}+CS_{5,1,1,3}+CS_{5,1,1,1}) \\ 2SS_{5,1,1,2}-SS_{5,1,1,4} \\ 4(CC_{1,1,1,1}-CC_{1,1,1,3}-2CC_{3,1,1,1}+2CC_{3,1,1,3}+CC_{5,1,1,1}-CC_{5,1,1,3}) \\ 4(SS_{3,1,1,2}-SS_{5,1,1,2}) \end{bmatrix}$$

$$\sin^3\theta\sin 2\phi B = \begin{bmatrix} \sin^4\theta\cos\phi\sin 2\phi \\ \sin^4\theta\sin\phi\sin 2\phi \\ \sin^3\theta\cos\theta\sin 2\phi \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 4(SS_{4,1,1,3}+SS_{4,1,1,1}) \\ 4(SC_{4,1,1,1}-SC_{4,1,1,3}) \\ 2SS_{1,2,1,2}-SS_{1,4,1,2} \end{bmatrix}$$

M. PARTIALS OF W12

$W_{12} = SS_{3,1,2,1} = \int_0^{2\pi}\int_0^{\pi}\sin^3\theta\sin^2\phi\, g(R,\theta,\phi)\,d\theta\, d\phi$ so the partials are given by:

$$\sin^3\theta\sin^2\phi A_1 = \begin{bmatrix} \sin^5\theta\cos^2\phi\sin^2\phi \\ 2\sin^5\theta\cos\phi\sin^3\phi \\ 2\sin^4\theta\cos\theta\cos\phi\sin^2\phi \\ \sin^5\theta\sin^4\phi \\ 2\sin^4\theta\cos\theta\sin^3\phi \\ \cos^2\theta\sin^3\theta\sin^2\phi \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4(SS_{5,1,2,1}-SS_{5,1,4,1}) \\ 2SS_{5,1,1,2}-SS_{5,1,1,4} \\ 8(CC_{1,1,1,1}-CC_{1,1,3,1}-2CC_{3,1,1,1}+2CC_{3,1,3,1}+CC_{5,1,1,1}-CC_{5,1,3,1}) \\ 4SS_{5,1,4,1} \\ 8(CS_{1,1,3,1}-2CS_{3,1,3,1}+CS_{5,1,3,1}) \\ 4(SS_{3,1,2,1}-SS_{5,1,2,1}) \end{bmatrix}$$

$$\sin^3\theta\sin^2\phi B = \begin{bmatrix} \sin^4\theta\cos\phi\sin^2\phi \\ \sin^4\theta\sin^3\phi \\ \sin^3\theta\cos\theta\sin^2\phi \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8(SC_{4,1,1,1}-SC_{4,1,3,1}) \\ 8SS_{4,1,3,1} \\ 2SS_{1,2,2,1}-SS_{1,4,2,1} \end{bmatrix}$$

APPENDIX C

A. PARTIALS OF U1,U2 AND U3

This step simply involves taking derivatives of

$$[u_1 \;\; u_2 \;\; u_3]^T = \left[\frac{w_8}{w_5} \;\; \frac{w_{10}}{w_5} \;\; \frac{w_6}{2w_5}\right]^T$$

with respect to each parameter. The results are:

$$\frac{\partial u_1}{\partial w_5} = -\frac{w_8}{w_5^2}$$

$$\frac{\partial u_1}{\partial w_8} = \frac{1}{w_5}$$

$$\frac{\partial u_2}{\partial w_5} = -\frac{w_{10}}{w_5^2}$$

$$\frac{\partial u_2}{\partial w_{10}} = \frac{1}{w_5}$$

$$\frac{\partial u_3}{\partial w_5} = -\frac{w_6}{2w_5^2}$$

$$\frac{\partial u_3}{\partial w_6} = \frac{1}{2w_5}$$

and all other partials = 0.

B. PARTIALS OF U4,..,U9 WRT MU

$$\frac{\partial u_4}{\partial \mu} = -\frac{w_8}{2w_5}\begin{bmatrix}\sigma^{11}\\ \sigma^{12}\\ \sigma^{13}\end{bmatrix}$$

$$\frac{\partial u_5}{\partial \mu} = -\frac{w_{10}}{w_5}\begin{bmatrix}\sigma^{11}\\ \sigma^{12}\\ \sigma^{13}\end{bmatrix}$$

$$\frac{\partial u_6}{\partial \mu} = -\frac{w_6}{2w_5}\begin{bmatrix}\sigma^{11}\\ \sigma^{12}\\ \sigma^{13}\end{bmatrix}$$

$$\frac{\partial u_7}{\partial \mu} = -\frac{w_{10}}{2w_5}\begin{bmatrix}\sigma^{12}\\ \sigma^{22}\\ \sigma^{23}\end{bmatrix}$$

$$\frac{\partial u_8}{\partial \mu} = -\frac{w_6}{2w_5}\begin{bmatrix}\sigma^{12}\\ \sigma^{22}\\ \sigma^{23}\end{bmatrix}$$

$$\frac{\partial u_9}{\partial \mu} = -\frac{w_6}{4w_5}\begin{bmatrix}\sigma^{13}\\ \sigma^{23}\\ \sigma^{33}\end{bmatrix}$$

C. PARTIALS OF U4,..,U9 WRT SIGMA

To solve for these partials we use the table of partials given in the previous section and use standard calculus rules:

1. Partials of U4

$$\frac{\partial u_4}{\partial \sigma_{11}} = -\sigma^{11} u_4$$

$$\frac{\partial u_4}{\partial \sigma_{12}} = -2\sigma^{12} u_4 - \frac{1}{q^2}\left[\sigma_{33}\left(\frac{Rw_{11}}{4w_5} - \frac{\mu_2 w_8}{2w_5}\right) - \sigma_{23}\left(\frac{R(w_1 - w_2)}{8w_5} - \frac{\mu_3 w_8}{2w_5}\right)\right]$$

$$\frac{\partial u_4}{\partial \sigma_{13}} = -2\sigma^{13} u_4 - \frac{1}{q^2}\left[\sigma_{22}\left(\frac{R(w_1 - w_2)}{8w_5} - \frac{\mu_3 w_8}{2w_5}\right) - \sigma_{23}\left(\frac{Rw_{11}}{4w_5} - \frac{\mu_2 w_8}{2w_5}\right)\right]$$

$$\frac{\partial u_4}{\partial \sigma_{22}} = -\sigma^{22} u_4 - \frac{1}{q^2}\left[\sigma_{13}\left(\frac{R(w_1 - w_2)}{8w_5} - \frac{\mu_3 w_8}{2w_5}\right) - \sigma_{33}\left(\frac{Rw_9}{2w_5} - \frac{\mu_1 w_8}{2w_5}\right)\right]$$

$$\frac{\partial u_4}{\partial \sigma_{23}} = -2\sigma^{23} u_4 - \frac{1}{q^2}\left[\sigma_{23}\left(\frac{Rw_9 - \mu_1 w_8}{w_5}\right)\right.$$
$$\left. - \sigma_{13}\left(\frac{Rw_{11}}{4w_5} - \frac{\mu_2 w_8}{2w_5}\right) - \sigma_{12}\left(\frac{R(w_1 - w_2)}{8w_5} - \frac{\mu_3 w_8}{2w_5}\right)\right]$$

$$\frac{\partial u_4}{\partial \sigma_{33}} = -\sigma^{33} u_4 - \frac{1}{q^2}\left[\sigma_{12}\left(\frac{Rw_{11}}{4w_5} - \frac{\mu_2 w_8}{2w_5}\right) - \sigma_{22}\left(\frac{Rw_9}{2w_5} - \frac{\mu_1 w_8}{2w_5}\right)\right]$$

2. Partials of U5

$$\frac{\partial u_5}{\partial \sigma_{11}} = -\sigma^{11} u_5$$

$$\frac{\partial u_5}{\partial \sigma_{12}} = -2\sigma^{12} u_5 - \frac{1}{q^2}\left[\sigma_{33}\left(\frac{Rw_{12}}{w_5} - \frac{\mu_2 w_{10}}{w_5}\right) - \sigma_{23}\left(\frac{R(w_3 - w_4)}{4w_5} - \frac{\mu_3 w_{10}}{w_5}\right)\right]$$

$$\frac{\partial u_5}{\partial \sigma_{13}} = -2\sigma^{13} u_5 - \frac{1}{q^2}\left[\sigma_{22}\left(\frac{R(w_3 - w_4)}{4w_5} - \frac{\mu_3 w_{10}}{w_5}\right) - \sigma_{23}\left(\frac{Rw_{12}}{w_5} - \frac{\mu_2 w_{10}}{w_5}\right)\right]$$

$$\frac{\partial u_5}{\partial \sigma_{22}} = -\sigma^{22} u_5 - \frac{1}{q^2}\left[\sigma_{13}\left(\frac{R(w_3 - w_4)}{4w_5} - \frac{\mu_3 w_{10}}{w_5}\right) - \sigma_{33}\left(\frac{Rw_{11}}{2w_5} - \frac{\mu_1 w_{10}}{w_5}\right)\right]$$

$$\frac{\partial u_5}{\partial \sigma_{23}} = -2\sigma^{23} u_5 - \frac{1}{q^2}\left[\sigma_{23}\left(\frac{Rw_{11} - 2\mu_1 w_{10}}{w_5}\right)\right.$$
$$\left. - \sigma_{13}\left(\frac{Rw_{12} - \mu_2 w_{10}}{w_5}\right) - \sigma_{12}\left(\frac{R(w_3 - w_4)}{4w_5} - \frac{\mu_3 w_{10}}{w_5}\right)\right]$$

$$\frac{\partial u_5}{\partial \sigma_{33}} = -\sigma^{33} u_5 - \frac{1}{q^2}\left[\sigma_{12}\left(\frac{Rw_{12}}{w_5} - \frac{\mu_2 w_{10}}{w_5}\right) - \sigma_{22}\left(\frac{Rw_{11}}{2w_5} - \frac{\mu_1 w_{10}}{w_5}\right)\right]$$

3. Partials of U6

$$\frac{\partial u_6}{\partial \sigma_{11}} = -\sigma^{11} u_6$$

$$\frac{\partial u_6}{\partial \sigma_{12}} = -2\sigma^{12} u_6 - \frac{1}{q^2}\left[\sigma_{33}\left(\frac{R(w_3 - w_4)}{4w_5} - \frac{\mu_2 w_6}{2w_5}\right) - \sigma_{23}\left(\frac{R(w_7 + w_5)}{4w_5} - \frac{\mu_3 w_6}{2w_5}\right)\right]$$

$$\frac{\partial u_6}{\partial \sigma_{13}} = -2\sigma^{13}u_6 - \frac{1}{q^2}\left[\sigma_{22}\left(\frac{R(w_7+w_5)}{4w_5} - \frac{\mu_3 w_6}{2w_5}\right) - \sigma_{23}\left(\frac{R(w_3-w_4)}{4w_5} - \frac{\mu_2 w_6}{2w_5}\right)\right]$$

$$\frac{\partial u_6}{\partial \sigma_{22}} = -\sigma^{22}u_6 - \frac{1}{q^2}\left[\sigma_{13}\left(\frac{R(w_7+w_5)}{4w_5} - \frac{\mu_3 w_6}{2w_5}\right) - \sigma_{33}\left(\frac{R(w_1-w_2)}{4w_5} - \frac{\mu_1 w_6}{2w_5}\right)\right]$$

$$\frac{\partial u_6}{\partial \sigma_{23}} = -2\sigma^{23}u_6 - \frac{1}{q^2}\left[\sigma_{23}\left(\frac{R(w_1-w_2)}{2w_5} - \frac{\mu_1 w_6}{w_5}\right)\right.$$

$$\left. -\sigma_{13}\left(\frac{R(w_3-w_4)}{4w_5} - \frac{\mu_2 w_6}{2w_5}\right) - \sigma_{12}\left(\frac{R(w_7+w_5)}{4w_5} - \frac{\mu_3 w_6}{2w_5}\right)\right]$$

$$\frac{\partial u_6}{\partial \sigma_{33}} = -\sigma^{33}u_6 - \frac{1}{q^2}\left[\sigma_{12}\left(\frac{R(w_3-w_4)}{4w_5} - \frac{\mu_2 w_6}{2w_5}\right) - \sigma_{22}\left(\frac{R(w_1-w_2)}{4w_5} - \frac{\mu_1 w_6}{2w_5}\right)\right]$$

4. Partials of U7

$$\frac{\partial u_7}{\partial \sigma_{11}} = -\sigma^{11}u_7 - \frac{1}{q^2}\left[\sigma_{23}\left(\frac{R(w_3-w_4)}{8w_5} - \frac{\mu_3 w_{10}}{2w_5}\right) - \sigma_{33}\left(\frac{Rw_{12}}{2w_5} - \frac{\mu_2 w_{10}}{2w_5}\right)\right]$$

$$\frac{\partial u_7}{\partial \sigma_{12}} = -2\sigma^{12}u_7 - \frac{1}{q^2}\left[\sigma_{33}\left(\frac{Rw_{11}}{4w_5} - \frac{\mu_1 w_{10}}{2w_5}\right) - \sigma_{13}\left(\frac{R(w_3-w_4)}{8w_5} - \frac{\mu_3 w_{10}}{2w_5}\right)\right]$$

$$\frac{\partial u_7}{\partial \sigma_{13}} = -2\sigma^{13}u_7 - \frac{1}{q^2}\left[\sigma_{13}\left(\frac{Rw_{12}-\mu_2 w_{10}}{w_5}\right)\right.$$

$$\left. -\sigma_{23}\left(\frac{Rw_{11}}{4w_5} - \frac{\mu_1 w_{10}}{2w_5}\right) - \sigma_{12}\left(\frac{R(w_3-w_4)}{8w_7} - \frac{\mu_3 w_{10}}{2w_5}\right)\right]$$

$$\frac{\partial u_7}{\partial \sigma_{22}} = -\sigma^{22} u_7$$

$$\frac{\partial u_7}{\partial \sigma_{23}} = -2\sigma^{23} u_7 - \frac{1}{q^2}\left[\sigma_{11}\left(\frac{R(w_3 - w_4)}{8w_5} - \frac{\mu_3 w_{10}}{2w_5}\right) - \sigma_{13}\left(\frac{Rw_{11}}{4w_5} - \frac{\mu_1 w_{10}}{2w_5}\right)\right]$$

$$\frac{\partial u_7}{\partial \sigma_{33}} = -\sigma^{33} u_7 - \frac{1}{q^2}\left[\sigma_{12}\left(\frac{Rw_{11}}{4w_5} - \frac{\mu_1 w_{10}}{2w_5}\right) - \sigma_{11}\left(\frac{Rw_{12}}{2w_5} - \frac{\mu_2 w_{10}}{2w_5}\right)\right]$$

5. Partials of U8

$$\frac{\partial u_8}{\partial \sigma_{11}} = -\sigma^{11} u_8 - \frac{1}{q^2}\left[\sigma_{23}\left(\frac{R(w_7 + w_5)}{4w_5} - \frac{\mu_3 w_6}{2w_5}\right) - \sigma_{33}\left(\frac{R(w_3 - w_4)}{4w_5} - \frac{\mu 2 w_6}{2w_5}\right)\right]$$

$$\frac{\partial u_8}{\partial \sigma_{12}} = -2\sigma^{12} u_8 - \frac{1}{q^2}\left[\sigma_{33}\left(\frac{R(w_1 - w_2)}{4w_5} - \frac{\mu_1 w_6}{2w_5}\right) - \sigma_{13}\left(\frac{R(w_7 + w_5)}{4w_5} - \frac{\mu_3 w_6}{2w_5}\right)\right]$$

$$\frac{\partial u_8}{\partial \sigma_{13}} = -2\sigma^{13} u_8 - \frac{1}{q^2}\left[\sigma_{13}\left(\frac{R(w_3 - w_4)}{2w_5} - \frac{\mu_2 w_6}{w_5}\right)\right.$$
$$\left. - \sigma_{23}\left(\frac{R(w_1 - w_2)}{4w_5} - \frac{\mu_1 w_6}{2w_5}\right) - \sigma_{12}\left(\frac{R(w_7 + w_5)}{4w_5} - \frac{\mu_3 w_6}{2w_5}\right)\right]$$

$$\frac{\partial u_8}{\partial \sigma_{22}} = -\sigma^{22} u_8$$

$$\frac{\partial u_8}{\partial \sigma_{23}} = -2\sigma^{23} u_8 - \frac{1}{q^2}\left[\sigma_{11}\left(\frac{R(w_7 + w_5)}{4w_5} - \frac{\mu_3 w_6}{2w_5}\right) - \sigma_{13}\left(\frac{R(w_1 - w_2)}{4w_5} - \frac{\mu_1 w_6}{2w_5}\right)\right]$$

$$\frac{\partial u_8}{\partial \sigma_{33}} = -\sigma^{33}u_8 - \frac{1}{q^2}\left[\sigma_{12}\left(\frac{R(w_1-w_2)}{4w_5} - \frac{\mu_1 w_6}{2w_5}\right) - \sigma_{11}\left(\frac{R(w_3-w_4)}{4w_5} - \frac{\mu_2 w_6}{2w_5}\right)\right]$$

6. Partials of U9

$$\frac{\partial u_9}{\partial \sigma_{11}} = -\sigma^{11}u_9 - \frac{1}{q^2}\left[\sigma_{23}\left(\frac{R(w_3-w_4)}{8w_5} - \frac{\mu_2 w_6}{4w_5}\right) - \sigma_{22}\left(\frac{R(w_7+w_5)}{8w_5} - \frac{\mu_3 w_6}{4w_5}\right)\right]$$

$$\frac{\partial u_9}{\partial \sigma_{12}} = -2\sigma^{12}u_9 - \frac{1}{q^2}\left[\sigma_{12}\left(\frac{R(w_7+w_5)}{4w_5} - \frac{\mu_3 w_6}{2w_5}\right)\right.$$
$$\left. - \sigma_{13}\left(\frac{R(w_3-w_4)}{8w_5} - \frac{\mu_2 w_6}{4w_5}\right) - \sigma_{23}\left(\frac{R(w_1-w_2)}{8w_5} - \frac{\mu_1 w_6}{4w_5}\right)\right]$$

$$\frac{\partial u_9}{\partial \sigma_{13}} = -2\sigma^{13}u_9 - \frac{1}{q^2}\left[\sigma_{22}\left(\frac{R(w_1-w_2)}{8w_5} - \frac{\mu_1 w_6}{4w_5}\right) - \sigma_{12}\left(\frac{R(w_3-w_4)}{8w_5} - \frac{\mu_2 w_6}{4w_5}\right)\right]$$

$$\frac{\partial u_9}{\partial \sigma_{22}} = -\sigma^{22}u_9 - \frac{1}{q^2}\left[\sigma_{13}\left(\frac{R(w_1-w_2)}{8w_5} - \frac{\mu_1 w_6}{4w_5}\right) - \sigma_{11}\left(\frac{R(w_7+w_5)}{8w_5} - \frac{\mu_3 w_6}{4w_5}\right)\right]$$

$$\frac{\partial u_9}{\partial \sigma_{23}} = -2\sigma^{23}u_9 - \frac{1}{q^2}\left[\sigma_{11}\left(\frac{R(w_3-w_4)}{8w_5} - \frac{\mu_2 w_6}{4w_5}\right) - \sigma_{12}\left(\frac{R(w_1-w_2)}{8w_5} - \frac{\mu_1 w_6}{4w_5}\right)\right]$$

$$\frac{\partial u_9}{\partial \sigma_{33}} = -\sigma^{33}u_9$$

D. PARTIALS OF U4,..,U9 WRT R

$$\frac{\partial u_4}{\partial R} = \frac{1}{8w_5}\left(4\sigma^{11}w_9 + 2\sigma^{12}w_{11} + \sigma^{13}(w_1 - w_2)\right)$$

$$\frac{\partial u_5}{\partial R} = \frac{1}{4w_5}\left(2\sigma^{11}w_{11} + 4\sigma^{12}w_{12} + \sigma^{13}(w_3 - w_4)\right)$$

$$\frac{\partial u_6}{\partial R} = \frac{1}{4w_5}\left(\sigma^{11}(w_1 - w_2) + \sigma^{12}(w_3 - w_4) + \sigma^{13}(w_7 + w_5)\right)$$

$$\frac{\partial u_7}{\partial R} = \frac{1}{8w_5}\left(2\sigma^{12}w_{11} + 4\sigma^{22}w_{12} + \sigma^{23}(w_3 - w_4)\right)$$

$$\frac{\partial u_8}{\partial R} = \frac{1}{4w_5}\left(\sigma^{12}(w_1 - w_2) + \sigma^{22}(w_3 - w_4) + \sigma^{23}(w_7 + w_5)\right)$$

$$\frac{\partial u_9}{\partial R} = \frac{1}{8w_5}\left(\sigma^{13}(w_1 - w_2) + \sigma^{23}(w_3 - w_4) + \sigma^{33}(w_7 + w_5)\right)$$

E. PARTIALS OF U4,..,U9 WRT W

1. Partials of U4

$$\frac{\partial u_4}{\partial w_1} = \frac{R\sigma^{13}}{8w_5}$$

$$\frac{\partial u_4}{\partial w_2} = -\frac{R\sigma^{13}}{8w_5}$$

$$\frac{\partial u_4}{\partial w_5} = -\frac{1}{w_5} u_4$$

$$\frac{\partial u_4}{\partial w_8} = -\frac{1}{2 w_5}\left(\sigma^{11}\mu_1+\sigma^{12}\mu_2+\sigma^{13}\mu_3\right)$$

$$\frac{\partial u_4}{\partial w_9} = \frac{R\sigma^{11}}{2 w_5}$$

$$\frac{\partial u_4}{\partial w_{11}} = \frac{R\sigma^{12}}{4 w_5}$$

2. Partials of U5

$$\frac{\partial u_5}{\partial w_3} = \frac{R\sigma^{13}}{4 w_5}$$

$$\frac{\partial u_5}{\partial w_4} = -\frac{R\sigma^{13}}{4 w_5}$$

$$\frac{\partial u_5}{\partial w_5} = -\frac{1}{w_5} u_5$$

$$\frac{\partial u_5}{\partial w_{10}} = -\frac{1}{w_5}\left(\sigma^{11}\mu_1+\sigma^{12}\mu_2+\sigma^{13}\mu_3\right)$$

$$\frac{\partial u_5}{\partial w_{11}} = \frac{R\sigma^{11}}{2 w_5}$$

$$\frac{\partial u_5}{\partial w_{12}} = \frac{R\sigma^{12}}{w_5}$$

3. Partials of U6

$$\frac{\partial u_6}{\partial w_1} = \frac{R\sigma^{11}}{4w_5}$$

$$\frac{\partial u_6}{\partial w_2} = -\frac{R\sigma^{11}}{4w_5}$$

$$\frac{\partial u_6}{\partial w_3} = \frac{R\sigma^{12}}{4w_5}$$

$$\frac{\partial u_6}{\partial w_4} = -\frac{R\sigma^{12}}{4w_5}$$

$$\frac{\partial u_6}{\partial w_5} = -\frac{1}{w_5}u_6 + \frac{R\sigma^{13}}{4w_5}$$

$$\frac{\partial u_6}{\partial w_6} = -\frac{1}{2w_5}\left(\sigma^{11}\mu_1 + \sigma^{12}\mu_2 + \sigma^{13}\mu_3\right)$$

$$\frac{\partial u_6}{\partial w_7} = \frac{R\sigma^{13}}{4w_5}$$

4. Partials of U7

$$\frac{\partial u_7}{\partial w_3} = \frac{R\sigma^{23}}{8w_5}$$

$$\frac{\partial u_7}{\partial w_4} = -\frac{R\sigma^{23}}{8w_5}$$

$$\frac{\partial u_7}{\partial w_5} = -\frac{1}{w_5}u_7$$

$$\frac{\partial u_7}{\partial w_{10}} = -\frac{1}{2w_5}\left(\sigma^{12}\mu_1+\sigma^{22}\mu_2+\sigma^{23}\mu_3\right)$$

$$\frac{\partial u_7}{\partial w_{11}} = \frac{R\sigma^{12}}{4w_5}$$

$$\frac{\partial u_7}{\partial w_{12}} = \frac{R\sigma^{22}}{2w_5}$$

5. Partials of U8

$$\frac{\partial u_8}{\partial w_1} = \frac{R\sigma^{12}}{4w_5}$$

$$\frac{\partial u_8}{\partial w_2} = -\frac{R\sigma^{12}}{4w_5}$$

$$\frac{\partial u_8}{\partial w_3} = \frac{R\sigma^{22}}{4w_5}$$

$$\frac{\partial u_8}{\partial w_4} = -\frac{R\sigma^{22}}{4w_5}$$

$$\frac{\partial u_8}{\partial w_5} = -\frac{1}{w_5}u_8 + \frac{R\sigma^{23}}{4w_5}$$

$$\frac{\partial u_8}{\partial w_6} = -\frac{1}{2w_5}\left(\sigma^{12}\mu_1 + \sigma^{22}\mu_2 + \sigma^{23}\mu_3\right)$$

$$\frac{\partial u_8}{\partial w_7} = \frac{R\sigma^{23}}{4w_5}$$

6. Partials of U9

$$\frac{\partial u_9}{\partial w_1} = \frac{R\sigma^{13}}{8w_5}$$

$$\frac{\partial u_9}{\partial w_2} = -\frac{R\sigma^{13}}{8w_5}$$

$$\frac{\partial u_9}{\partial w_3} = \frac{R\sigma^{23}}{8w_5}$$

$$\frac{\partial u_9}{\partial w_4} = -\frac{R\sigma^{23}}{8w_5}$$

$$\frac{\partial u_9}{\partial w_5} = -\frac{1}{w_5}u_9 + \frac{R\sigma^{33}}{8w_5}$$

$$\frac{\partial u_9}{\partial w_6} = -\frac{1}{4w_5}\left(\sigma^{13}\mu_1+\sigma^{23}\mu_2+\sigma^{33}\mu_3\right)$$

$$\frac{\partial u_9}{\partial w_7} = \frac{R\sigma^{33}}{8w_5}$$

APPENDIX D

```
      PROGRAM SEPCMP

*                 PROGRAMMER:    LT ARTHUR F. BROCK
*                 DATE:          12 MAY 1991
*                 LAST MODIFIED: 29 AUGUST 1991
*        PROGRAM DETERMINES FIRST AND SECOND ORDER APPROXIMATIONS TO
*     VARIANCE OF SEP GIVEN VALUES OF MU AND SIGMA (VARIANCE-COVARIANCE
*     MATRIX) AS INPUTS.  OTHER INPUTS ARE PARAMETERS FOR USE IN THE
*     IMSL SUBROUTINE QTWODQ USED TO PERFORM GAUSSIAN QUADRATURE
*     EVALUATION OF DOUBLE INTEGRALS.

      INCLUDE 'COM DEF'
      INCLUDE 'WVEC DEF'
      INCLUDE 'PMUCOM DEF'
      INTEGER I,OT1,OT2,SAMPSZ,K
      REAL DT1(10,9),DT2(22,10),DT3(9,22),SEPM(9,9),PMU(3,3),PSIG(6,6)
     C       ,SEPMU(3),SEPSIG(6),CI(2),MPT(3,3)
      CALL INIT(SAMPSZ,MPT)
      WRITE(16,51) R,NUMTR
         WRITE(16,52)(MU(I),I=1,3)
      DO 20 I=1,3
         WRITE(16,53)(MSIG(I,J),J=1,3)
   20 CONTINUE
      OT1=0
      OT2=0
      DO 10 I=1,SAMPSZ
         CALL INIT2(MPT)
         CALL WVALS
         CALL DT1VAL(DT1,SEPMU,SEPSIG)
         CALL DT2VAL(DT2)
         CALL DT3VAL(DT3)
         CALL SEPEST(DT1,DT2,DT3,SEPM)
         CALL PMUS(MSIG,PMU,PSIG)
         CALL BNDEST(SEPM,PMU,PSIG,SEPMU,SEPSIG,CI)
         IF(SEPSET .LT. (R-CI(1)) .OR. SEPSET .GT. (R+CI(1))) THEN
            OT1=OT1+1
            IF(SEPSET .LT.(R-CI(2)).OR. SEPSET .GT.(R+CI(2))) THEN
               OT2=OT2+1
            ENDIF
         ENDIF
         PRINT*,'I = ',I,REAL(I-OT1)/REAL(I),REAL(I-OT2)/REAL(I)
         WRITE(16,50) R-CI(1),R+CI(1),OT1,R-CI(2),R+CI(2),OT2
   10 CONTINUE
      WRITE(16,55) SAMPSZ,REAL(OT1)/REAL(SAMPSZ),REAL(OT2)/REAL(SAMPSZ)
   50 FORMAT(1X,2('  (',F7.2,',',F7.2,')',2X,I3,2X))
   51 FORMAT(1X,'TEST FOR SEP = ',F7.2,/,1X,'SAMPLES PER TRIAL = ',I2)
   52 FORMAT(1X,'          MU ',/,3(F7.2,','),/,/,1X,
     C       '                  SIGMA')
   53 FORMAT(1X,3(F9.3,3X))
   55 FORMAT(1X,/,'SAMPLE SIZE: ',I4,3X,2(F5.3,8X))
      STOP
      END
```

```
      SUBROUTINE INIT(SAMPSZ,MPT)
*       INITIALIZE FILES, READ INPUT DATA AND INITIALIZE VARIABLES.
      INCLUDE 'COM DEF'
      INCLUDE 'PMUCOM DEF'
      INTEGER N,SAMPSZ,RK
      REAL MPT(3,3),TOL,NUMSIG(3,3)
      EXTERNAL RNSET,DCHFAC
      OPEN(15,FILE='/MUSIG DATA A1')
      OPEN(16,FILE='/OUTDT DATA A1')
      READ(15,*) MU(1)
      READ(15,*) MU(2)
      READ(15,*) MU(3)
      READ(15,*) SIG(1)
      READ(15,*) SIG(2)
      READ(15,*) SIG(3)
      READ(15,*) SIG(4)
      READ(15,*) SIG(5)
      READ(15,*) SIG(6)
      READ(15,*) ERRABS
      READ(15,*) ERRREL
      READ(15,*) IRULE
      READ(15,*) NUMTR
      READ(15,*) SAMPSZ
      READ(15,*) ISEED
      READ(15,*) NUMSIM
      CALL RNSET(ISEED+INT(MU(1)+SIG(1)))
      DET=SIG(1)*(SIG(4)*SIG(6)-(SIG(5)**2))-SIG(2)*(SIG(2)*SIG(6)
     C        -SIG(3)*SIG(5))+SIG(3)*(SIG(2)*SIG(5)-SIG(3)*SIG(4))
      IF (DET .LE. 0.0000000001) THEN
         WRITE(16,50) DET
         STOP
      ENDIF
      SIGINV(1)=(SIG(4)*SIG(6)-(SIG(5)**2))/DET
      SIGINV(2)=(SIG(3)*SIG(5)-SIG(2)*SIG(6))/DET
      SIGINV(3)=(SIG(2)*SIG(5)-SIG(3)*SIG(4))/DET
      SIGINV(4)=(SIG(1)*SIG(6)-(SIG(3)**2))/DET
      SIGINV(5)=(SIG(3)*SIG(2)-SIG(1)*SIG(5))/DET
      SIGINV(6)=(SIG(1)*SIG(4)-(SIG(2)**2))/DET
      DENOM=(SQRT(2.0*PI)**3)*SQRT(DET)
      MSIG(1,1)=SIG(1)
      MSIG(1,2)=SIG(2)
      MSIG(1,3)=SIG(3)
      MSIG(2,1)=SIG(2)
      MSIG(2,2)=SIG(4)
      MSIG(2,3)=SIG(5)
      MSIG(3,1)=SIG(3)
      MSIG(3,2)=SIG(5)
      MSIG(3,3)=SIG(6)
      CALL FINDSEP
      MSIG(1,1)=SIG(1)/NUMSIM
      MSIG(1,2)=SIG(2)/NUMSIM
      MSIG(1,3)=SIG(3)/NUMSIM
      MSIG(2,1)=SIG(2)/NUMSIM
      MSIG(2,2)=SIG(4)/NUMSIM
      MSIG(2,3)=SIG(5)/NUMSIM
      MSIG(3,1)=SIG(3)/NUMSIM
      MSIG(3,2)=SIG(5)/NUMSIM
      MSIG(3,3)=SIG(6)/NUMSIM
      TOL=100.0*DMACH(4)
      CALL DCHFAC(3,MSIG,3,TOL,RK,MPT,3)
      CALL SETDAT
      CALL MAKES
 50   FORMAT(1X,'ILLEGAL INPUT MATRIX, DETERMINANT = ',F8.5,' PROGRAM',
     C        ' TERMINATED.')
      RETURN
      END
```

```
      SUBROUTINE INIT2(MPT)

*       INITIALIZE VALUES USED EACH TIME THROUGH LOOP.

      REAL MPT(3,3),TMU(3),TSIG(6)
      INCLUDE 'COM DEF'
      INCLUDE 'PMUCOM DEF'
 20   CONTINUE
         SIG(1)=0.0
         SIG(2)=0.0
         SIG(3)=0.0
         SIG(4)=0.0
         SIG(5)=0.0
         SIG(6)=0.0
         MU(1)=0.0
         MU(2)=0.0
         MU(3)=0.0
         DO 15 I=1,INT(NUMSIM)
            CALL TRIALS(NUMTR,MPT,TMU,TSIG)
            SIG(1)=SIG(1)+TSIG(1)
            SIG(2)=SIG(2)+TSIG(2)
            SIG(3)=SIG(3)+TSIG(3)
            SIG(4)=SIG(4)+TSIG(4)
            SIG(5)=SIG(5)+TSIG(5)
            SIG(6)=SIG(6)+TSIG(6)
            MU(1)=MU(1)+TMU(1)
            MU(2)=MU(2)+TMU(2)
            MU(3)=MU(3)+TMU(3)
 15      CONTINUE
         MSIG(1,1)=SIG(1)
         MSIG(1,2)=SIG(2)
         MSIG(1,3)=SIG(3)
         MSIG(2,1)=SIG(2)
         MSIG(2,2)=SIG(4)
         MSIG(2,3)=SIG(5)
         MSIG(3,1)=SIG(3)
         MSIG(3,2)=SIG(5)
         MSIG(3,3)=SIG(6)
         DET=SIG(1)*(SIG(4)*SIG(6)-(SIG(5)**2))-SIG(2)*(SIG(2)*SIG(6)
     C          -SIG(3)*SIG(5))+SIG(3)*(SIG(2)*SIG(5)-SIG(3)*SIG(4))
      IF (DET .GT. 0.00000000001) GOTO 30
      GOTO 20
 30   CONTINUE
      SIGINV(1)=(SIG(4)*SIG(6)-(SIG(5)**2))/DET
      SIGINV(2)=(SIG(3)*SIG(5)-SIG(2)*SIG(6))/DET
      SIGINV(3)=(SIG(2)*SIG(5)-SIG(3)*SIG(4))/DET
      SIGINV(4)=(SIG(1)*SIG(6)-(SIG(3)**2))/DET
      SIGINV(5)=(SIG(3)*SIG(2)-SIG(1)*SIG(5))/DET
      SIGINV(6)=(SIG(1)*SIG(4)-(SIG(2)**2))/DET
      DENOM=(SQRT(2.0*PI)**3)*SQRT(DET)
      CALL FINDSEP
      RETURN
      END


      SUBROUTINE MXTMLT(M1,M2,ANS,I,J,K)

*        PERFORMS MATRIX MULTIPLICATION M1(I,J)*M2(J,K) TO PRODUCE
*        OUTPUT MATRIX ANS(I,K)

      INTEGER L,ROW,COL,I,J,K
      REAL M1(I,J),M2(J,K),ANS(I,K)
      DO 10 ROW=1,I
         DO 20 COL=1,K
            ANS(ROW,COL)=0.0
 20      CONTINUE
 10   CONTINUE
      DO 30 ROW=1,I
```

```
         DO 40 COL=1,K
            DO 50 L=1,J
               ANS(ROW,COL)=ANS(ROW,COL)+M1(ROW,L)*M2(L,COL)
   50       CONTINUE
   40    CONTINUE
   30 CONTINUE
      RETURN
      END


      SUBROUTINE SEPEST(DT1,DT2,DT3,SEPM)

*        PRODUCES THE 9 BY 9 MATRIX OF PARTIALS OF SEP

      INCLUDE 'COM DEF'
      REAL DT1(10,9),DT2(22,10),DT3(9,22),SEPM(9,9),TEMP(22,9)
      CALL MXTMLT(DT2,DT1,TEMP,22,10,9)
      CALL MXTMLT(DT3,TEMP,SEPM,9,22,9)
      RETURN
      END


      SUBROUTINE SETDAT

*       SETS THE INPUT VALUES OF MU, SIG AND SEP TO BE HELD CONSTANT
*       THROUGHOUT THE LOOPS OF THE PROGRAM.

      INCLUDE 'COM DEF'
      INCLUDE 'PMUCOM DEF'
      INTEGER I,J
      DO10 I=1,3
         MUSET(I)=MU(I)
   10 CONTINUE
      SEPSET=R
      RETURN
      END


      SUBROUTINE TRIALS(N,MPT,TMU,TSIG)

*        DRAWS N RANDOM SAMPLES FROM NORMAL(MU,SIGMA) AND DETERMINES
*        ESTIMATES FOR MU AND SIGMA BASED ON THESE SAMPLES.

      INCLUDE 'COM DEF'
      INCLUDE 'PMUCOM DEF'
      INTEGER I,N,J
      REAL MPT(3,3),RVAR(50,3),RSQ(50,3),RMLT(50,3),RMN(6),OP(3)
     C     ,NUMFAC,TSIG(6),TMU(3)
      EXTERNAL DRNMVN,RNSET,DMACH,DRNUNF
      DO 10 I=1,N
         CALL DRNMVN(1,3,MPT,3,OP,1)
         RVAR(I,1)=OP(1)+MUSET(1)/NUMSIM
         RVAR(I,2)=OP(2)+MUSET(2)/NUMSIM
         RVAR(I,3)=OP(3)+MUSET(3)/NUMSIM
         RSQ(I,1)=RVAR(I,1)**2
         RSQ(I,2)=RVAR(I,2)**2
         RSQ(I,3)=RVAR(I,3)**2
         RMLT(I,1)=RVAR(I,1)*RVAR(I,2)
         RMLT(I,2)=RVAR(I,1)*RVAR(I,3)
         RMLT(I,3)=RVAR(I,2)*RVAR(I,3)
   10 CONTINUE
      TMU(1)=FMN(RVAR,1,N)
      TMU(2)=FMN(RVAR,2,N)
      TMU(3)=FMN(RVAR,3,N)
      RMN(1)=FMN(RSQ,1,N)
      RMN(2)=FMN(RSQ,2,N)
      RMN(3)=FMN(RSQ,3,N)
      RMN(4)=FMN(RMLT,1,N)
```

```
      RMN(5)=FMN(RMLT,2,N)
      RMN(6)=FMN(RMLT,3,N)
      NUMFAC=1.0
      TSIG(1)=(RMN(1)-TMU(1)**2)*NUMFAC
      TSIG(2)=(RMN(4)-TMU(1)*TMU(2))*NUMFAC
      TSIG(3)=(RMN(5)-TMU(1)*TMU(3))*NUMFAC
      TSIG(4)=(RMN(2)-TMU(2)**2)*NUMFAC
      TSIG(5)=(RMN(6)-TMU(2)*TMU(3))*NUMFAC
      TSIG(6)=(RMN(3)-TMU(3)**2)*NUMFAC
      RETURN
      END


      REAL FUNCTION FMN(VEC,COL,N)

*        DETERMINES THE MEAN OF N ITEMS IN COLUMN COL OF VECTOR VEC

      INTEGER I,N,COL
      REAL VEC(50,3),TOT
      TOT=0.0
      DO 10 I=1,N
         TOT=TOT+VEC(I,COL)
   10 CONTINUE
      FMN=TOT/REAL(N)
      RETURN
      END


      REAL FUNCTION G(X)
      REAL X
      G=0.0
      RETURN
      END


      REAL FUNCTION H(X)
      REAL X
      H=6.2831854
      RETURN
      END


      SUBROUTINE WVALS

*        DETERMINES VALUES OF W, CC, CS, SC AND SS FUNCTIONS USING THE
*        IMSL SUBROUTINE QTWODQ.

      REAL G,H,H1,H2,ERREST
      INCLUDE 'COM DEF'
      INCLUDE 'WVEC DEF'
      EXTERNAL G,H,DTWODQ,CC1111,CC1311,CC3111,CS1111,CS1311,CS3111,
     C        SC1100,SC1200,SC1300,SC2111,SC3121,SS2111,SS3112,SS3121,
     C        CC1113,CC1131,CC1211,CC1331,CC1411,CC1511,CC1531,CC2111,
     C        CC3113,CC3131,CC5111,CC5113,CC5131,CS1113,CS1131,CS1211,
     C        CS1331,CS1411,CS1511,CS1531,CS2111,CS3113,CS3131,CS5111,
     C        CS5113,CS5131,SC1121,SC1212,SC1221,SC1321,SC1400,SC1412,
     C        SC1421,SC1500,SC1521,SC3100,SC4111,SC4113,SC4131,SC5112,
     C        SC5114,SC5121,SC5141,SS1112,SS1121,SS1212,SS1221,SS1312,
     C        SS1321,SS1412,SS1421,SS1512,SS1521,SS4111,SS4113,SS4131,
     C        SS5112,SS5114,SS5121,SS5141
      B=PI
      CALL DTWODQ (CC1111,A,B,G,H,ERRABS,ERRREL,IRULE,W(1),ERREST)
      CALL DTWODQ (CC1311,A,B,G,H,ERRABS,ERRREL,IRULE,W(2),ERREST)
```

```
CALL DTWODQ (CS1111,A,B,G,H,ERRABS,ERRREL,IRULE,W(3),ERREST)
CALL DTWODQ (CS1311,A,B,G,H,ERRABS,ERRREL,IRULE,W(4),ERREST)
CALL DTWODQ (SC1100,A,B,G,H,ERRABS,ERRREL,IRULE,W(5),ERREST)
CALL DTWODQ (SC1200,A,B,G,H,ERRABS,ERRREL,IRULE,W(6),ERREST)
CALL DTWODQ (SC1300,A,B,G,H,ERRABS,ERRREL,IRULE,W(7),ERREST)
CALL DTWODQ (SC2111,A,B,G,H,ERRABS,ERRREL,IRULE,W(8),ERREST)
CALL DTWODQ (SC3121,A,B,G,H,ERRABS,ERRREL,IRULE,W(9),ERREST)
CALL DTWODQ (SS2111,A,B,G,H,ERRABS,ERRREL,IRULE,W(10),ERREST)
CALL DTWODQ (SS3112,A,B,G,H,ERRABS,ERRREL,IRULE,W(11),ERREST)
CALL DTWODQ (SS3121,A,B,G,H,ERRABS,ERRREL,IRULE,W(12),ERREST)
CALL DTWODQ (CC1113,A,B,G,H,ERRABS,ERRREL,IRULE,CC(1),ERREST)
CALL DTWODQ (CC1131,A,B,G,H,ERRABS,ERRREL,IRULE,CC(2),ERREST)
CALL DTWODQ (CC1211,A,B,G,H,ERRABS,ERRREL,IRULE,CC(3),ERREST)
CALL DTWODQ (CC1331,A,B,G,H,ERRABS,ERRREL,IRULE,CC(5),ERREST)
CALL DTWODQ (CC1411,A,B,G,H,ERRABS,ERRREL,IRULE,CC(6),ERREST)
CALL DTWODQ (CC1511,A,B,G,H,ERRABS,ERRREL,IRULE,CC(8),ERREST)
CALL DTWODQ (CC1531,A,B,G,H,ERRABS,ERRREL,IRULE,CC(9),ERREST)
CALL DTWODQ (CC2111,A,B,G,H,ERRABS,ERRREL,IRULE,CC(10),ERREST)
CALL DTWODQ (CC3111,A,B,G,H,ERRABS,ERRREL,IRULE,CC(17),ERREST)
CALL DTWODQ (CC3113,A,B,G,H,ERRABS,ERRREL,IRULE,CC(11),ERREST)
CALL DTWODQ (CC3131,A,B,G,H,ERRABS,ERRREL,IRULE,CC(12),ERREST)
CALL DTWODQ (CC5111,A,B,G,H,ERRABS,ERRREL,IRULE,CC(14),ERREST)
CALL DTWODQ (CC5113,A,B,G,H,ERRABS,ERRREL,IRULE,CC(15),ERREST)
CALL DTWODQ (CC5131,A,B,G,H,ERRABS,ERRREL,IRULE,CC(16),ERREST)
CALL DTWODQ (CS1113,A,B,G,H,ERRABS,ERRREL,IRULE,CS(1),ERREST)
CALL DTWODQ (CS1131,A,B,G,H,ERRABS,ERRREL,IRULE,CS(2),ERREST)
CALL DTWODQ (CS1211,A,B,G,H,ERRABS,ERRREL,IRULE,CS(3),ERREST)
CALL DTWODQ (CS1331,A,B,G,H,ERRABS,ERRREL,IRULE,CS(5),ERREST)
CALL DTWODQ (CS1411,A,B,G,H,ERRABS,ERRREL,IRULE,CS(6),ERREST)
CALL DTWODQ (CS1511,A,B,G,H,ERRABS,ERRREL,IRULE,CS(8),ERREST)
CALL DTWODQ (CS1531,A,B,G,H,ERRABS,ERRREL,IRULE,CS(9),ERREST)
CALL DTWODQ (CS2111,A,B,G,H,ERRABS,ERRREL,IRULE,CS(10),ERREST)
CALL DTWODQ (CS3111,A,B,G,H,ERRABS,ERRREL,IRULE,CS(17),ERREST)
CALL DTWODQ (CS3113,A,B,G,H,ERRABS,ERRREL,IRULE,CS(18),ERREST)
CALL DTWODQ (CS3131,A,B,G,H,ERRABS,ERRREL,IRULE,CS(11),ERREST)
CALL DTWODQ (CS5111,A,B,G,H,ERRABS,ERRREL,IRULE,CS(13),ERREST)
CALL DTWODQ (CS5113,A,B,G,H,ERRABS,ERRREL,IRULE,CS(14),ERREST)
CALL DTWODQ (CS5131,A,B,G,H,ERRABS,ERRREL,IRULE,CS(16),ERREST)
CALL DTWODQ (SC1121,A,B,G,H,ERRABS,ERRREL,IRULE,SC(1),ERREST)
CALL DTWODQ (SC1212,A,B,G,H,ERRABS,ERRREL,IRULE,SC(18),ERREST)
CALL DTWODQ (SC1221,A,B,G,H,ERRABS,ERRREL,IRULE,SC(2),ERREST)
CALL DTWODQ (SC1321,A,B,G,H,ERRABS,ERRREL,IRULE,SC(3),ERREST)
CALL DTWODQ (SC1400,A,B,G,H,ERRABS,ERRREL,IRULE,SC(4),ERREST)
CALL DTWODQ (SC1412,A,B,G,H,ERRABS,ERRREL,IRULE,SC(19),ERREST)
CALL DTWODQ (SC1421,A,B,G,H,ERRABS,ERRREL,IRULE,SC(5),ERREST)
CALL DTWODQ (SC1500,A,B,G,H,ERRABS,ERRREL,IRULE,SC(6),ERREST)
CALL DTWODQ (SC1521,A,B,G,H,ERRABS,ERRREL,IRULE,SC(7),ERREST)
CALL DTWODQ (SC3100,A,B,G,H,ERRABS,ERRREL,IRULE,SC(9),ERREST)
CALL DTWODQ (SC4111,A,B,G,H,ERRABS,ERRREL,IRULE,SC(10),ERREST)
CALL DTWODQ (SC4113,A,B,G,H,ERRABS,ERRREL,IRULE,SC(11),ERREST)
CALL DTWODQ (SC4131,A,B,G,H,ERRABS,ERRREL,IRULE,SC(12),ERREST)
CALL DTWODQ (SC5112,A,B,G,H,ERRABS,ERRREL,IRULE,SC(13),ERREST)
CALL DTWODQ (SC5114,A,B,G,H,ERRABS,ERRREL,IRULE,SC(14),ERREST)
CALL DTWODQ (SC5121,A,B,G,H,ERRABS,ERRREL,IRULE,SC(15),ERREST)
CALL DTWODQ (SC5141,A,B,G,H,ERRABS,ERRREL,IRULE,SC(16),ERREST)
CALL DTWODQ (SS1112,A,B,G,H,ERRABS,ERRREL,IRULE,SS(1),ERREST)
CALL DTWODQ (SS1121,A,B,G,H,ERRABS,ERRREL,IRULE,SS(2),ERREST)
CALL DTWODQ (SS1212,A,B,G,H,ERRABS,ERRREL,IRULE,SS(3),ERREST)
CALL DTWODQ (SS1221,A,B,G,H,ERRABS,ERRREL,IRULE,SS(4),ERREST)
CALL DTWODQ (SS1312,A,B,G,H,ERRABS,ERRREL,IRULE,SS(5),ERREST)
CALL DTWODQ (SS1321,A,B,G,H,ERRABS,ERRREL,IRULE,SS(6),ERREST)
CALL DTWODQ (SS1412,A,B,G,H,ERRABS,ERRREL,IRULE,SS(7),ERREST)
CALL DTWODQ (SS1421,A,B,G,H,ERRABS,ERRREL,IRULE,SS(8),ERREST)
CALL DTWODQ (SS1512,A,B,G,H,ERRABS,ERRREL,IRULE,SS(9),ERREST)
CALL DTWODQ (SS1521,A,B,G,H,ERRABS,ERRREL,IRULE,SS(10),ERREST)
CALL DTWODQ (SS4111,A,B,G,H,ERRABS,ERRREL,IRULE,SS(12),ERREST)
CALL DTWODQ (SS4113,A,B,G,H,ERRABS,ERRREL,IRULE,SS(13),ERREST)
CALL DTWODQ (SS4131,A,B,G,H,ERRABS,ERRREL,IRULE,SS(14),ERREST)
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      CALL DTWODQ (SS5112,A,B,G,H,ERRABS,ERRREL,IRULE,SS(15),ERREST)
      CALL DTWODQ (SS5114,A,B,G,H,ERRABS,ERRREL,IRULE,SS(16),ERREST)
      CALL DTWODQ (SS5121,A,B,G,H,ERRABS,ERRREL,IRULE,SS(17),ERREST)
      CALL DTWODQ (SS5141,A,B,G,H,ERRABS,ERRREL,IRULE,SS(18),ERREST)
      RETURN
      END

      SUBROUTINE FINDSEP

*   BASED ON PL-1 PROGRAM RAP, PRODUCES SEP BASED ON MU AND SIGMA
*   FINDSEP DIAGONALIZES SIGMA BASED ON EIGENVALUES AND THEN CALLS
*   SUBROUTINE ITERATION, WHICH DETERMINES SEP

      INCLUDE 'COM DEF'
      REAL EIGVLS(3),EIGMAT(3,3),Q1(3,3),NMEAN(3),NEWSIG(3,3)
     C    ,TEMP(3,3),EIGNML(3,3)
      INTEGER I,J
      EXTERNAL DEVCSF
      CALL DEVCSF(3,MSIG,3,EIGVLS,EIGMAT,3)
      CALL ORTHO(EIGMAT,EIGNML)
      DO 11 I=1,3
         DO 21 J=1,3
            Q1(I,J)=EIGNML(J,I)
  21     CONTINUE
  11  CONTINUE
      CALL MXTMLT(Q1,MU,NMEAN,3,3,1)
      CALL MXTMLT(MSIG,EIGNML,TEMP,3,3,3)
      CALL MXTMLT(Q1,TEMP,NEWSIG,3,3,3)
      CALL ITERATION(NMEAN,NEWSIG)
      RETURN
      END


      SUBROUTINE ORTHO (A1,A2)

*   ORTHONORMALIZES MATRIX OF EIGENVECTORS OF SIGMA

      REAL A1(3,3),A2(3,3),L,TEMP,TEMP2
      INTEGER I
      L=SQRT((A1(1,1)**2)+(A1(2,1)**2)+(A1(3,1)**2))
      DO 10 I=1,3
         A2(I,1)=A1(I,1)/L
  10  CONTINUE
      TEMP=A2(1,1)*A1(1,2)+A2(2,1)*A1(2,2)+A2(3,1)*A1(3,2)
      DO 20 I=1,3
         A2(I,2)=A1(I,2)-TEMP*A2(I,1)
  20  CONTINUE
      L=SQRT(A2(1,2)**2+A2(2,2)**2+A2(3,2)**2)
      DO 30 I=1,3
         A2(I,2)=A2(I,2)/L
  30  CONTINUE
      TEMP=A2(1,1)*A1(1,3)+A2(2,1)*A1(2,3)+A2(3,1)*A1(3,3)
      TEMP2=A2(1,2)*A1(1,3)+A2(2,2)*A1(2,3)+A2(3,2)*A1(3,3)
      DO 40 I=1,3
         A2(I,3)=A1(I,3)-TEMP*A2(I,1)-TEMP2*A2(I,2)
  40  CONTINUE
      L=SQRT(A2(1,3)**2+A2(2,3)**2+A2(3,3)**2)
      DO 50 I=1,3
         A2(I,3)=A2(I,3)/L
  50  CONTINUE
      RETURN
      END
```

```
      SUBROUTINE ITERATION(MEAN,SIGMA)

*   GENERATES SEP BASED ON MU AND SIGMA FOR DIAGONALIZED SIGMA

      INCLUDE 'COM DEF'
      INCLUDE 'SEPCOM DEF'
      REAL TOL,MEAN(3),SIGMA(3,3),MOM1,TRACE(3),SQSIG(3,3),CUSIG(3,3)
     C      ,TEMP(3),C2,DOF,RLOW,TPLOW,RHIGH,TPHIGH,CRATIO,INTEGRAL
     C      ,MOM2,MOM3
      TOL=0.001
      CXX=1.0/SIGMA(1,1)
      CYY=1.0/SIGMA(2,2)
      CZZ=1.0/SIGMA(3,3)
      CSANT=QUPITW*SQRT(CXX*CYY*CZZ)
      C=(CXX*(MEAN(1)**2)+CYY*(MEAN(2)**2)+CZZ*(MEAN(3)**2))/2.0
      CALL MXTMLT(MEAN,MEAN,MOM1,1,3,1)
      CALL MXTMLT(SIGMA,SIGMA,SQSIG,3,3,3)
      CALL MXTMLT(SIGMA,MEAN,TEMP,3,3,1)
      CALL MXTMLT(MEAN,TEMP,MOM2,1,3,1)
      CALL MXTMLT(SIGMA,SQSIG,CUSIG,3,3,3)
      CALL MXTMLT(SQSIG,MEAN,TEMP,3,3,1)
      CALL MXTMLT(MEAN,TEMP,MOM3,1,3,1)
      DO 10 I=1,3
         TRACE(I)=0.0
         MN(I)=MEAN(I)
 10   CONTINUE
      DO 20 I=1,3
         TRACE(1)=TRACE(1)+SIGMA(I,I)
         TRACE(2)=TRACE(2)+SQSIG(I,I)
         TRACE(3)=TRACE(3)+CUSIG(I,I)
 20   CONTINUE
      MOM1=MOM1+TRACE(1)
      MOM2=2.0*(TRACE(2)+2.0*MOM2)
      MOM3=8.0*(TRACE(3)+3.0*MOM3)
      BETA=(MOM3**2)/(MOM2**3)
      DOF=8.0/BETA
      C2=DOF*(1.0-(2.0/(9.0*DOF)))**3
      RADIUS=SQRT(ABS(SQRT(ABS(MOM2/(2.0*DOF)))*(C2-DOF)+MOM1))
      RLOW=0.95*RADIUS
      TPLOW=EVALT(RLOW)
 40   CONTINUE
      IF(TPLOW .GT. 0.5) THEN
         RLOW=0.9*RLOW
         TPLOW=EVALT(RLOW)
      GOTO 40
      ENDIF
      RHIGH=1.105*RLOW
      TPHIGH=EVALT(RHIGH)
 50   CONTINUE
      IF (TPHIGH .LT. 0.5) THEN
         RHIGH=1.1*RHIGH
         TPHIGH=EVALT(RHIGH)
      GOTO 50
      ENDIF
      CRATIO=(0.5-TPLOW)/(TPHIGH-TPLOW)
      RADIUS=RLOW+(RHIGH-RLOW)*CRATIO
      ICNT=0
 60   CONTINUE
      ICNT=ICNT+1
      IF((RHIGH-RLOW)/RHIGH .GT. TOL) THEN
         INTEGRAL=EVALT(RADIUS)
         IF(ABS(INTEGRAL - 0.5) .LT. TOL) GOTO 75
         IF(INTEGRAL .GT. 0.5) THEN
            RHIGH = RADIUS
            TPHIGH=INTEGRAL
         ELSE
            RLOW=RADIUS
            TPLOW=INTEGRAL
```

```
      ENDIF
      RADIUS=RLOW+(RHIGH-RLOW)*CRATIO
      IF (ICNT .GE. 150) THEN
         PRINT*,'COUNT EXCEEDED'
         GOTO 75
      ENDIF
    GOTO 60
    ENDIF
 75 CONTINUE
    R=RADIUS
    RETURN
    END


    REAL FUNCTION EVALT(RAD)

*   EVALUATES THE AREA COVERAGE BY TRIVARIATE NORMAL FOR INPUT R

    INCLUDE 'COM DEF'
    INCLUDE 'SEPCOM DEF'
    REAL ANS,LOWER,UPPER,RAD
    EXTERNAL RAPFC3,RAPLOW,RAPUP
    RADIUS=RAD
    LOWER=0.0
    UPPER=6.2831853
    CALL DTWODQ(RAPFC3,LOWER,UPPER,RAPLOW,RAPUP,ERRABS,ERRREL,IRULE,
   C            ANS,ERREST)

    EVALT=ANS
    RETURN
    END


    REAL FUNCTION RAPLOW(X)
    REAL X
    RAPLOW=0.0
    RETURN
    END


    REAL FUNCTION RAPUP(X)
    REAL X
    RAPUP=3.1415927
    RETURN
    END


    REAL FUNCTION RAPFC3(THETA,PHI)

*   FUNCTION USED TO FIND SEP

    REAL CP,SP,CT,ST,AA,SA,SAM1,BB,B2DA,ERFARG,EXPARG,THETA,PHI
    INCLUDE 'COM DEF'
    INCLUDE 'SEPCOM DEF'
    EXTERNAL ERF
    CP=COS(PHI)
    CT=COS(THETA)
    SP=SIN(PHI)
    ST=SIN(THETA)
    AA=(CXX*SP*SP*CT*CT+CYY*SP*SP*ST*ST+CZZ*CP*CP)/2.0
    SA=SQRT(AA)
    SAM1=1.0/SA
    BB=(CXX*MN(1)*SP*CT+CYY*MN(2)*SP*ST+CZZ*MN(3)*CP)/2.0
    B2DA=BB*BB/AA
    ERFARG=SA*RADIUS+SAM1*BB
    EXPARG=-AA*RADIUS*RADIUS-2.0*BB*RADIUS-B2DA
    IF (EXPARG .GT. -32.0) THEN
       F=ERF(ERFARG)*SAM1*(0.5+B2DA)-EXP(EXPARG)*(RADIUS-BB/AA)
```

```
C         *SQPIINV
      ELSE
         F=ERF(ERFARG)*SAM1*(0.5+B2DA)
      ENDIF
      ERFARG=SAM1*BB
      EXPARG=-B2DA
      IF (EXPARG .GT. -32.0) THEN
         F=F-ERF(ERFARG)*SAM1*(0.5+B2DA)-EXP(EXPARG)*(BB/AA)*SQPIINV
      ELSE
         F=F-ERF(ERFARG)*SAM1*(0.5+B2DA)
      ENDIF
      EXPARG=B2DA-C
      IF (EXPARG .GT. -32.0) THEN
         F=F*CSANT*EXP(EXPARG)*SP/AA
         IF(F .LT. 0.000000000000001) F=0.0
      ELSE
         F=0.0
      ENDIF
      RAPFC3=F
      RETURN
      END


      SUBROUTINE DT1VAL (DT1,SEPMU,SEPSIG)

*          SETS THE VALUES OF THE MATRIX DT1 AND THE VECTORS SEPMU AND
*          SEPSIG.

      INCLUDE 'COM DEF'
      INCLUDE 'WVEC DEF'
      REAL DT1(10,9),SEPMU(3),SEPSIG(6)
      INTEGER I,J
      DO 100 I=1,9
         DO 200 J=1,9
            IF (I .NE. J) THEN
               DT1(I,J)=0.0
            ELSE
               DT1(I,J)=1.0
            ENDIF
 200     CONTINUE
 100  CONTINUE
      DT1(10,1)=W(8)/W(5)
      DT1(10,2)=W(10)/W(5)
      DT1(10,3)=W(6)/(2.0*W(5))
      DT1(10,4)=(R/(8.0*W(5)))*(4.0*SIGINV(1)*W(9)+2.0*SIGINV(2)*W(11)
     C          +SIGINV(3)*(W(1)-W(2)))-(W(8)/(2.0*W(5)))*
     C          (SIGINV(1)*MU(1)+SIGINV(2)*MU(2)+SIGINV(3)*MU(3))
      DT1(10,5)=(R/(4.0*W(5)))*(2.0*SIGINV(1)*W(11)+4.0*SIGINV(2)*W(12)
     C          +SIGINV(3)*(W(3)-W(4)))-(W(10)/W(5))*
     C          (SIGINV(1)*MU(1)+SIGINV(2)*MU(2)+SIGINV(3)*MU(3))
      DT1(10,6)=(R/(4.0*W(5)))*(SIGINV(1)*(W(1)-W(2))+SIGINV(2)*(W(3)
     C          -W(4))+SIGINV(3)*(W(7)+W(5)))-(W(6)/(2.0*W(5)))*
     C          (SIGINV(1)*MU(1)+SIGINV(2)*MU(2)+SIGINV(3)*MU(3))
      DT1(10,7)=(R/(8.0*W(5)))*(2.0*SIGINV(2)*W(11)+4.0*SIGINV(4)
     C          *W(12)+SIGINV(5)*(W(3)-W(4)))-(W(10)/(2.0*W(5)))*
     C          (SIGINV(2)*MU(1)+SIGINV(4)*MU(2)+SIGINV(5)*MU(3))
      DT1(10,8)=(R/(4.0*W(5)))*(SIGINV(2)*(W(1)-W(2))+SIGINV(4)*(W(3)
     C          -W(4))+SIGINV(5)*(W(7)+W(5)))-(W(6)/(2.0*W(5)))*
     C          (SIGINV(2)*MU(1)+SIGINV(4)*MU(2)+SIGINV(5)*MU(3))
      DT1(10,9)=(R/(8.0*W(5)))*(SIGINV(3)*(W(1)-W(2))+SIGINV(5)*(W(3)
     C          -W(4))+SIGINV(6)*(W(7)+W(5)))-(W(6)/(4.0*W(5)))*
     C          (SIGINV(3)*MU(1)+SIGINV(5)*MU(2)+SIGINV(6)*MU(3))
      DO 10 I=1,3
         SEPMU(I)=DT1(10,I)
 10   CONTINUE
      DO 20 I=1,6
         SEPSIG(I)=DT1(10,I+3)
```

```
   20 CONTINUE
      RETURN
      END


      SUBROUTINE DT2VAL(DT2)

*        SETS THE VALUES OF THE MATRIX DT2.  THE VECTORS A1 AND B1 ARE
*        IN THIS SUBROUTINE AND THEN PASSED TO THE SUBROUTINE FIXMAT,
*        WHICH ACTUALLY SETS THE VALUES OF DT2.

      INCLUDE 'COM DEF'
      INCLUDE 'WVEC DEF'
      REAL DT2(22,10),SIGM(3,3),TEMP(3),ANS(3),B1(3),A1(6)
      INTEGER I,J
      DO 100 I=1,10
         DO 200 J=1,10
            IF (I .NE. J) THEN
               DT2(I,J)=0.0
            ELSE
               DT2(I,J)=1.0
            ENDIF
  200    CONTINUE
  100 CONTINUE
      SIGM(1,1)=SIGINV(1)
      SIGM(1,2)=SIGINV(2)
      SIGM(1,3)=SIGINV(3)
      SIGM(2,1)=SIGINV(2)
      SIGM(2,2)=SIGINV(4)
      SIGM(2,3)=SIGINV(5)
      SIGM(3,1)=SIGINV(3)
      SIGM(3,2)=SIGINV(5)
      SIGM(3,3)=SIGINV(6)
         B1(1)=SC(2)/2.0
         B1(2)=SS(3)/4.0
         B1(3)=CC(10)
         A1(1)=CC(2)-CC(12)
         A1(2)=2.0*(W(3)-CS(17)-CS(2)+CS(11))
         A1(3)=2.0*(SC(1)-W(9))
         A1(4)=W(1)-CC(17)-CC(2)+CC(12)
         A1(5)=SS(1)-W(11)
         A1(6)=CC(17)
      CALL FIXMAT(A1,B1,11,DT2,SIGM)
         B1(1)=(SC(5)-SC(2))/2.0
         B1(2)=(SS(7)-SS(3))/4.0
         B1(3)=(CC(6)+CC(3))/2.0
         A1(1)=(2.0*CC(5)-CC(9)-CC(2))/4.0
         A1(2)=(2.0*W(4)-CS(8)-W(3)-2.0*CS(5)+CS(9)+CS(2))/2.0
         A1(3)=(SC(7)-SC(1))/2.0
         A1(4)=(2.0*W(2)-CC(8)-W(1)-2.0*CC(5)+CC(9)+CC(2))/4.0
         A1(5)=(SS(9)-SS(1))/4.0
         A1(6)=(CC(8)+2.0*W(2)+W(1))/4.0
      CALL FIXMAT(A1,B1,12,DT2,SIGM)
         B1(1)=SS(3)/4.0
         B1(2)=SS(4)/2.0
         B1(3)=CS(10)
         A1(1)=W(3)-CS(17)-CS(2)+CS(11)
         A1(2)=2.0*(W(1)-CC(2)-CC(17)+CC(12))
         A1(3)=SS(1)-W(11)
         A1(4)=CS(2)-CS(11)
         A1(5)=2.0*(SS(2)-W(12))
         A1(6)=CS(17)
      CALL FIXMAT(A1,B1,13,DT2,SIGM)
         B1(1)=(SS(7)-SS(3))/4.0
         B1(2)=(SS(8)-SS(4))/2.0
         B1(3)=(CS(6)+CS(3))/2.0
         A1(1)=(2.0*W(4)-2.0*CS(5)-CS(8)+CS(9)-W(3)+CS(2))/4.0
         A1(2)=(2.0*W(2)-2.0*CC(5)-CC(8)+CC(9)-W(1)+CC(2))/2.0
```

```
      A1(3)=(SS(9)-SS(1))/4.0
      A1(4)=(2.0*CS(5)-CS(9)-CS(2))/4.0
      A1(5)=(SS(10)-SS(2))/2.0
      A1(6)=(CS(8)+2.0*W(4)+W(3))/4.0
   CALL FIXMAT(A1,B1,14,DT2,SIGM)
      B1(1)=W(8)
      B1(2)=W(10)
      B1(3)=W(6)/2.0
      A1(1)=W(9)
      A1(2)=W(11)
      A1(3)=2.0*(W(1)-CC(17))
      A1(4)=W(12)
      A1(5)=2.0*(W(3)-CS(17))
      A1(6)=W(5)-SC(9)
   CALL FIXMAT(A1,B1,15,DT2,SIGM)
      B1(1)=(W(1)-W(2))/2.0
      B1(2)=(W(3)-W(4))/2.0
      B1(3)=(W(7)+W(5))/2.0
      A1(1)=(2.0*SC(2)-SC(5))/4.0
      A1(2)=(2.0*SS(3)-SS(7))/4.0
      A1(3)=(2.0*W(8)+CC(3)-CC(6))/2.0
      A1(4)=(2.0*SS(4)-SS(8))/4.0
      A1(5)=(2.0*W(10)+CS(3)-CS(6))/2.0
      A1(6)=(2.0*W(6)+SC(4))/4.0
   CALL FIXMAT(A1,B1,16,DT2,SIGM)
      B1(1)=(CC(3)-CC(6))/2.0
      B1(2)=(CS(3)-CS(6))/2.0
      B1(3)=(SC(4)+W(6))/2.0
      A1(1)=(2.0*SC(3)-SC(1)-SC(7))/4.0
      A1(2)=(2.0*SS(5)-SS(1)-SS(9))/4.0
      A1(3)=(W(1)-CC(8))/2.0
      A1(4)=(2.0*SS(6)-SS(2)-SS(10))/4.0
      A1(5)=(W(3)-CS(8))/2.0
      A1(6)=(SC(6)+2.0*W(7)+W(5))/4.0
   CALL FIXMAT(A1,B1,17,DT2,SIGM)
      B1(1)=W(9)
      B1(2)=W(11)/2.0
      B1(3)=W(1)-CC(17)
      A1(1)=SC(12)
      A1(2)=2.0*(SS(12)-SS(14))
      A1(3)=(2.0*SC(2)-SC(5))/4.0
      A1(4)=SC(10)-SC(12)
      A1(5)=(2.0*SS(3)-SS(7))/8.0
      A1(6)=W(8)-SC(10)
   CALL FIXMAT(A1,B1,18,DT2,SIGM)
      B1(1)=SC(12)
      B1(2)=SS(12)-SS(14)
      B1(3)=(2.0*SC(2)-SC(5))/8.0
      A1(1)=SC(16)
      A1(2)=(SS(16)+2.0*SS(15))/4.0
      A1(3)=2.0*(CC(2)-2.0*CC(12)+CC(16))
      A1(4)=SS(17)-SS(18)
      A1(5)=2.0*(W(3)-CS(2)-2.0*CS(17)+2.0*CS(11)+CS(13)-CS(16))
      A1(6)=W(9)-SC(15)
   CALL FIXMAT(A1,B1,19,DT2,SIGM)
      B1(1)=W(11)/2.0
      B1(2)=W(12)
      B1(3)=W(3)-CS(17)
      A1(1)=SS(12)-SS(14)
      A1(2)=2.0*(SC(10)-SC(12))
      A1(3)=(2.0*SS(3)-SS(7))/8.0
      A1(4)=SS(14)
      A1(5)=(2.0*SS(4)-SS(8))/4.0
      A1(6)=W(10)-SS(12)
   CALL FIXMAT(A1,B1,20,DT2,SIGM)
      B1(1)=(SS(13)+SS(12))/2.0
      B1(2)=(SC(10)-SC(11))/2.0
      B1(3)=(2.0*SS(3)-SS(7))/8.0
```

```
      A1(1)=(SS(16)+2.0*SS(15))/4.0
      A1(2)=(2.0*SC(15)-SC(14)-SC(13))/2.0
      A1(3)=CS(1)+W(3)-2.0*CS(18)-2.0*CS(17)+CS(14)+CS(13)
      A1(4)=(2.0*SS(15)-SS(16))/4.0
      A1(5)=W(1)-CC(1)-2.0*CC(17)+2.0*CC(11)+CC(14)-CC(15)
      A1(6)=W(11)-SS(15)
   CALL FIXMAT(A1,B1,21,DT2,SIGM)
      B1(1)=SC(10)-SC(12)
      B1(2)=SS(14)
      B1(3)=(2.0*SS(4)-SS(8))/8.0
      A1(1)=SS(17)-SS(18)
      A1(2)=(2.0*SS(15)-SS(16))/4.0
      A1(3)=2.0*(W(1)-CC(2)-2.0*CC(17)+2.0*CC(12)+CC(14)-CC(16))
      A1(4)=SS(18)
      A1(5)=2.0*(CS(2)-2.0*CS(11)+CS(16))
      A1(6)=W(12)-SS(17)
   CALL FIXMAT(A1,B1,22,DT2,SIGM)
   RETURN
   END


   SUBROUTINE FIXMAT(A1,B1,N,DT2,SIGM)

*        SETS THE VALUES OF DT2, GIVEN A1, B1, SIGMA INVERSE (SIGM)
*        AND THE ROW NUMBER TO BE SET AS INPUTS.

   INCLUDE 'WVEC DEF'
   INCLUDE 'COM DEF'
   REAL DT2(22,10),SIGM(3,3),A1(6),B1(3),A2(6),A3(6),A4(3),MSUM(6)
  C      ,G(6),ANS(3)
   INTEGER N,I
   DO 100 I=1,3
      A4(I)=B1(I)*R-MU(I)*W(N-10)
 100  CONTINUE
   CALL MXTMLT(SIGM,A4,ANS,3,3,1)
   DT2(N,1)=ANS(1)
   DT2(N,2)=ANS(2)
   DT2(N,3)=ANS(3)
   A2(1)=B1(1)*MU(1)
   A2(2)=B1(2)*MU(1)+B1(1)*MU(2)
   A2(3)=B1(3)*MU(1)+B1(1)*MU(3)
   A2(4)=B1(2)*MU(2)
   A2(5)=B1(3)*MU(2)+B1(2)*MU(3)
   A2(6)=B1(3)*MU(3)
   DO 200 I=1,6
      MSUM(I)=R*A1(I)-A2(I)
 200  CONTINUE
   CALL MXTMLT(SIGINV,MSUM,ANS,1,6,1)
   DT2(N,10)=-ANS(1)
   A3(1)=W(N-10)*(MU(1)**2)
   A3(2)=2.0*W(N-10)*MU(1)*MU(2)
   A3(3)=2.0*W(N-10)*MU(1)*MU(3)
   A3(4)=W(N-10)*(MU(2)**2)
   A3(5)=2.0*W(N-10)*MU(2)*MU(3)
   A3(6)=W(N-10)*(MU(3)**2)
   DO 300 I=1,6
      MSUM(I)=(R**2)*A1(I)-2.0*R*A2(I)+A3(I)
 300 CONTINUE
   G(1)=SIGINV(1)*SIGINV(1)
   G(2)=SIGINV(2)*SIGINV(1)
   G(3)=SIGINV(3)*SIGINV(1)
   G(4)=SIGINV(4)*SIGINV(1)-SIG(6)/DET
   G(5)=SIGINV(5)*SIGINV(1)+SIG(5)/DET
   G(6)=SIGINV(6)*SIGINV(1)-SIG(4)/DET
   CALL MXTMLT(G,MSUM,ANS,1,6,1)
   DT2(N,4)=(ANS(1)-W(N-10)*SIGINV(1))/2.0
   G(1)=SIGINV(1)*SIGINV(2)
   G(2)=SIGINV(2)*SIGINV(2)+SIG(6)/(2.0*DET)
```

```
      G(3)=SIGINV(3)*SIGINV(2)-SIG(5)/(2.0*DET)
      G(4)=SIGINV(4)*SIGINV(2)
      G(5)=SIGINV(5)*SIGINV(2)-SIG(3)/(2.0*DET)
      G(6)=SIGINV(6)*SIGINV(2)+SIG(2)/DET
      CALL MXTMLT(G,MSUM,ANS,1,6,1)
      DT2(N,5)=ANS(1)-W(N-10)*SIGINV(2)
      G(1)=SIGINV(1)*SIGINV(3)
      G(2)=SIGINV(2)*SIGINV(3)-SIG(5)/(2.0*DET)
      G(3)=SIGINV(3)*SIGINV(3)+SIG(4)/(2.0*DET)
      G(4)=SIGINV(4)*SIGINV(3)+SIG(3)/DET
      G(5)=SIGINV(5)*SIGINV(3)-SIG(2)/(2.0*DET)
      G(6)=SIGINV(6)*SIGINV(3)
      CALL MXTMLT(G,MSUM,ANS,1,6,1)
      DT2(N,6)=ANS(1)-W(N-10)*SIGINV(3)
      G(1)=SIGINV(1)*SIGINV(4)-SIG(6)/DET
      G(2)=SIGINV(2)*SIGINV(4)
      G(3)=SIGINV(3)*SIGINV(4)+SIG(3)/DET
      G(4)=SIGINV(4)*SIGINV(4)
      G(5)=SIGINV(5)*SIGINV(4)
      G(6)=SIGINV(6)*SIGINV(4)-SIG(1)/DET
      CALL MXTMLT(G,MSUM,ANS,1,6,1)
      DT2(N,7)=(ANS(1)-W(N-10)*SIGINV(4))/2.0
      G(1)=SIGINV(1)*SIGINV(5)+SIG(5)/DET
      G(2)=SIGINV(2)*SIGINV(5)-SIG(3)/(2.0*DET)
      G(3)=SIGINV(3)*SIGINV(5)-SIG(2)/(2.0*DET)
      G(4)=SIGINV(4)*SIGINV(5)
      G(5)=SIGINV(5)*SIGINV(5)+SIG(1)/(2.0*DET)
      G(6)=SIGINV(6)*SIGINV(5)
      CALL MXTMLT(G,MSUM,ANS,1,6,1)
      DT2(N,8)=ANS(1)-W(N-10)*SIGINV(5)
      G(1)=SIGINV(1)*SIGINV(6)-SIG(4)/DET
      G(2)=SIGINV(2)*SIGINV(6)+SIG(2)/DET
      G(3)=SIGINV(3)*SIGINV(6)
      G(4)=SIGINV(4)*SIGINV(6)-SIG(1)/DET
      G(5)=SIGINV(5)*SIGINV(6)
      G(6)=SIGINV(6)*SIGINV(6)
      CALL MXTMLT(G,MSUM,ANS,1,6,1)
      DT2(N,9)=(ANS(1)-W(N-10)*SIGINV(6))/2.0
      RETURN
      END


      SUBROUTINE DT3VAL(DT3)

*         SETS THE VALUES OF THE MATRIX DT3

      INCLUDE 'COM DEF'
      INCLUDE 'WVEC DEF'
      REAL DT3(9,22),U41,U42,U43,U51,U52,U61,U62,U71,U72,U73,U81,U82,
     C     U91,U92,U93
      INTEGER I,J
      DO 100 I=1,3
         DO 200 J=1,22
           DT3(I,J)=0.0
 200     CONTINUE
 100  CONTINUE
      DT3(1,15)=-W(8)/(W(5)**2)
      DT3(1,18)=1.0/W(5)
      DT3(2,15)=-W(10)/(W(5)**2)
      DT3(2,20)=1.0/W(5)
      DT3(3,15)=-0.5*W(6)/(W(5)**2)
      DT3(3,16)=0.5/W(5)
      U41=R*(2.0*SIGINV(1)*W(9)+SIGINV(2)*W(11)+0.5*SIGINV(3)*(W(1)
     C       -W(2)))/(4.0*W(5))
      U42=W(8)/(2.0*W(5))
      U43=SIGINV(1)*MU(1)+SIGINV(2)*MU(2)+SIGINV(3)*MU(3)
      U4=U41-U42*U43
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      DT3(4,1)=-SIGINV(1)*U42
      DT3(4,2)=-SIGINV(2)*U42
      DT3(4,3)=-SIGINV(3)*U42
      U51=R*(SIGINV(1)*W(11)+2.0*SIGINV(2)*W(12)+0.5*SIGINV(3)*(W(3)
     C      -W(4)))/(2.0*W(5))
      U52=W(10)/W(5)
      U5=U51-U52*U43
      DT3(5,1)=-SIGINV(1)*U52
      DT3(5,2)=-SIGINV(2)*U52
      DT3(5,3)=-SIGINV(3)*U52
      U61=R*(SIGINV(1)*(W(1)-W(2))+SIGINV(2)*(W(3)-W(4))+SIGINV(3)
     C      *(W(7)+W(5)))/(4.0*W(5))
      U62=W(6)/(2.0*W(5))
      U6=U61-U62*U43
      DT3(6,1)=-SIGINV(1)*U62
      DT3(6,2)=-SIGINV(2)*U62
      DT3(6,3)=-SIGINV(3)*U62
      U71=R*(SIGINV(2)*W(11)+2.0*SIGINV(4)*W(12)+0.5*SIGINV(5)
     C      *(W(3)-W(4)))/(4.0*W(5))
      U72=W(10)/(2.0*W(5))
      U73=SIGINV(2)*MU(1)+SIGINV(4)*MU(2)+SIGINV(5)*MU(3)
      U7=U71-U72*U73
      DT3(7,1)=-SIGINV(2)*U72
      DT3(7,2)=-SIGINV(4)*U72
      DT3(7,3)=-SIGINV(5)*U72
      U81=R*(SIGINV(2)*(W(1)-W(2))+SIGINV(4)*(W(3)-W(4))+SIGINV(5)
     C      *(W(7)+W(5)))/(4.0*W(5))
      U82=W(6)/(2.0*W(5))
      U8=U81-U82*U73
      DT3(8,1)=-SIGINV(2)*U82
      DT3(8,2)=-SIGINV(4)*U82
      DT3(8,3)=-SIGINV(5)*U82
      U91=R*(SIGINV(3)*(W(1)-W(2))+SIGINV(5)*(W(3)-W(4))+SIGINV(6)
     C      *(W(7)+W(5)))/(8.0*W(5))
      U92=W(6)/(4.0*W(5))
      U93=SIGINV(3)*MU(1)+SIGINV(5)*MU(2)+SIGINV(6)*MU(3)
      U9=U91-U92*U93
      DT3(9,1)=-SIGINV(3)*U92
      DT3(9,2)=-SIGINV(5)*U92
      DT3(9,3)=-SIGINV(6)*U92
      TEMP1=((R*W(11)/2.0)-MU(2)*W(8))/(2.0*W(5))
      TEMP2=(0.25*R*(W(1)-W(2))-MU(3)*W(8))/(2.0*W(5))
      TEMP3=(R*W(9)-MU(1)*W(8))/(2.0*W(5))
      DT3(4,4)=-SIGINV(1)*U4
      DT3(4,5)=-2.0*SIGINV(2)*U4-(SIG(6)*TEMP1-SIG(5)*TEMP2)/DET
      DT3(4,6)=-2.0*SIGINV(3)*U4-(SIG(4)*TEMP2-SIG(5)*TEMP1)/DET
      DT3(4,7)=-SIGINV(4)*U4-(SIG(3)*TEMP2-SIG(6)*TEMP3)/DET
      DT3(4,8)=-2.0*SIGINV(5)*U4-(2.0*SIG(5)*TEMP3-SIG(3)*TEMP1-SIG(2)
     C         *TEMP2)/DET
      DT3(4,9)=-SIGINV(6)*U4-(SIG(2)*TEMP1-SIG(4)*TEMP3)/DET
      TEMP1=(R*W(12)-MU(2)*W(10))/W(5)
      TEMP2=(0.25*R*(W(3)-W(4))-MU(3)*W(10))/W(5)
      TEMP3=(R*W(11)/2.0-MU(1)*W(10))/W(5)
      DT3(5,4)=-SIGINV(1)*U5
      DT3(5,5)=-2.0*SIGINV(2)*U5-(SIG(6)*TEMP1-SIG(5)*TEMP2)/DET
      DT3(5,6)=-2.0*SIGINV(3)*U5-(SIG(4)*TEMP2-SIG(5)*TEMP1)/DET
      DT3(5,7)=-SIGINV(4)*U5-(SIG(3)*TEMP2-SIG(6)*TEMP3)/DET
      DT3(5,8)=-2.0*SIGINV(5)*U5-(2.0*SIG(5)*TEMP3-SIG(3)*TEMP1-SIG(2)
     C         *TEMP2)/DET
      DT3(5,9)=-SIGINV(6)*U5-(SIG(2)*TEMP1-SIG(4)*TEMP3)/DET
      DT3(7,4)=-SIGINV(1)*U7-(SIG(5)*TEMP2-SIG(6)*TEMP1)/(2.0*DET)
      DT3(7,5)=-2.0*SIGINV(2)*U7-(SIG(6)*TEMP3-SIG(3)*TEMP2)/(2.0*DET)
      DT3(7,6)=-2.0*SIGINV(3)*U7-(2.0*SIG(3)*TEMP1-SIG(5)*TEMP3-SIG(2)
     C         *TEMP2)/(2.0*DET)
      DT3(7,7)=-SIGINV(4)*U7
      DT3(7,8)=-2.0*SIGINV(5)*U7-(SIG(1)*TEMP2-SIG(3)*TEMP3)/(2.0*DET)
      DT3(7,9)=-SIGINV(6)*U7-(SIG(2)*TEMP3-SIG(1)*TEMP1)/(2.0*DET)
      TEMP1=(R*(W(3)-W(4))/2.0-MU(2)*W(6))/(2.0*W(5))
```

```
      TEMP2=(R*(W(7)+W(5))/2.0-MU(3)*W(6))/(2.0*W(5))
      TEMP3=(R*(W(1)-W(2))/2.0-MU(1)*W(6))/(2.0*W(5))
      DT3(6,4)=-SIGINV(1)*U6
      DT3(6,5)=-2.0*SIGINV(2)*U6-(SIG(6)*TEMP1-SIG(5)*TEMP2)/DET
      DT3(6,6)=-2.0*SIGINV(3)*U6-(SIG(4)*TEMP2-SIG(5)*TEMP1)/DET
      DT3(6,7)=-SIGINV(4)*U6-(SIG(3)*TEMP2-SIG(6)*TEMP3)/DET
      DT3(6,8)=-2.0*SIGINV(5)*U6-(2.0*SIG(5)*TEMP3-SIG(3)*TEMP1-SIG(2)
     C          *TEMP2)/DET
      DT3(6,9)=-SIGINV(6)*U6-(SIG(2)*TEMP1-SIG(4)*TEMP3)/DET
      DT3(8,4)=-SIGINV(1)*U8-(SIG(5)*TEMP2-SIG(6)*TEMP1)/DET
      DT3(8,5)=-2.0*SIGINV(2)*U8-(SIG(6)*TEMP3-SIG(3)*TEMP2)/DET
      DT3(8,6)=-2.0*SIGINV(3)*U8-(2.0*SIG(3)*TEMP1-SIG(5)*TEMP3-SIG(2)
     C          *TEMP2)/DET
      DT3(8,7)=-SIGINV(4)*U8
      DT3(8,8)=-2.0*SIGINV(5)*U8-(SIG(1)*TEMP2-SIG(3)*TEMP3)/DET
      DT3(8,9)=-SIGINV(6)*U8-(SIG(2)*TEMP3-SIG(1)*TEMP1)/DET
      DT3(9,4)=-SIGINV(1)*U9-(SIG(5)*TEMP1-SIG(4)*TEMP2)/(2.0*DET)
      DT3(9,5)=-2.0*SIGINV(2)*U9-(2.0*SIG(2)*TEMP2-SIG(3)*TEMP1-SIG(5)
     C          *TEMP3)/(2.0*DET)
      DT3(9,6)=-2.0*SIGINV(3)*U9-(SIG(4)*TEMP3-SIG(2)*TEMP1)/(2.0*DET)
      DT3(9,7)=-SIGINV(4)*U9-(SIG(3)*TEMP3-SIG(1)*TEMP2)/(2.0*DET)
      DT3(9,8)=-2.0*SIGINV(5)*U9-(SIG(1)*TEMP1-SIG(2)*TEMP3)/(2.0*DET)
      DT3(9,9)=-SIGINV(6)*U9
      DT3(4,10)=U41/R
      DT3(5,10)=U51/R
      DT3(6,10)=U61/R
      DT3(7,10)=U71/R
      DT3(8,10)=U81/R
      DT3(9,10)=U91/R
      DO 300 I=4,9
         DO 400 J=11,22
            DT3(I,J)=0.0
400      CONTINUE
300   CONTINUE
      DT3(4,11)=(R*SIGINV(3))/(8.0*W(5))
      DT3(4,12)=-DT3(4,11)
      DT3(4,15)=-U4/W(5)
      DT3(4,18)=-U43/(2.0*W(5))
      DT3(4,19)=(R*SIGINV(1))/(2.0*W(5))
      DT3(4,21)=(R*SIGINV(2))/(4.0*W(5))
      DT3(5,13)=(R*SIGINV(3))/(4.0*W(5))
      DT3(5,14)=-DT3(5,13)
      DT3(5,15)=-U5/W(5)
      DT3(5,20)=-U43/W(5)
      DT3(5,21)=DT3(4,19)
      DT3(5,22)=(R*SIGINV(2))/W(5)
      DT3(6,11)=(R*SIGINV(1))/(4.0*W(5))
      DT3(6,12)=-DT3(6,11)
      DT3(6,13)=DT3(4,21)
      DT3(6,14)=-DT3(6,13)
      DT3(6,17)=(R*SIGINV(3))/(4.0*W(5))
      DT3(6,15)=-U6/W(5)+DT3(6,17)
      DT3(6,16)=-U43/(2.0*W(5))
      DT3(7,13)=(R*SIGINV(5))/(8.0*W(5))
      DT3(7,14)=-DT3(7,13)
      DT3(7,15)=-U7/W(5)
      DT3(7,20)=-U73/(2.0*W(5))
      DT3(7,21)=DT3(4,21)
      DT3(7,22)=(R*SIGINV(4))/(2.0*W(5))
      DT3(8,11)=DT3(4,21)
      DT3(8,12)=-DT3(4,21)
      DT3(8,13)=DT3(7,22)/2.0
      DT3(8,14)=-DT3(8,13)
      DT3(8,15)=-U8/W(5)+DT3(7,13)*2.0
      DT3(8,16)=DT3(7,20)
      DT3(8,17)=DT3(7,13)*2.0
      DT3(9,11)=DT3(4,11)
      DT3(9,12)=-DT3(9,11)
```

```
      DT3(9,13)=DT3(7,13)
      DT3(9,14)=-DT3(9,13)
      DT3(9,17)=(R*SIGINV(6))/(8.0*W(5))
      DT3(9,15)=-U9/W(5)+DT3(9,17)
      DT3(9,16)=-U93/(4.0*W(5))
      RETURN
      END


      SUBROUTINE MAKES

*   CREATES THE MATRICES S AND ST

      INCLUDE 'PMUCOM DEF'
      INTEGER I,J
      DO 10 I=1,9
         DO 20 J=1,6
            S(I,J)=0.0
 20      CONTINUE
 10   CONTINUE
      S(1,1)=1.0
      S(2,2)=0.5
      S(3,3)=0.5
      S(4,2)=0.5
      S(5,4)=1.0
      S(6,5)=0.5
      S(7,3)=0.5
      S(8,5)=0.5
      S(9,6)=1.0
      DO 30 I=1,6
         DO 40 J=1,9
            ST(I,J)=S(J,I)
 40      CONTINUE
 30   CONTINUE
      RETURN
      END


      SUBROUTINE PMUS(INP,OUTP1,OUTP2)

*   COMPUTES THE VALUES OF PMU AND PSIG

      INCLUDE 'PMUCOM DEF'
      INCLUDE 'COM DEF'
      REAL INP(3,3),OUTP1(3,3),PROD(9,9),TEMP(9,6),OUTP2(6,6)
      INTEGER I,J
      CALL TENSR(INP,INP,PROD,3,3,3,3)
      CALL MXTMLT(PROD,S,TEMP,9,9,6)
      CALL MXTMLT(ST,TEMP,OUTP2,6,9,6)
      DO 10 I=1,6
         DO 20 J=1,6
            OUTP2(I,J)=(2.0*OUTP2(I,J))/(REAL(NUMTR)*NUMSIM)
 20      CONTINUE
 10   CONTINUE
      DO 30 I=1,3
         DO 40 J=1,3
            OUTP1(I,J)=INP(I,J)/REAL(NUMTR)
 40      CONTINUE
 30   CONTINUE
      RETURN
      END
```

```
      SUBROUTINE TENSR(A,B,C,A1,A2,B1,B2)

*   COMPUTES THE TENSOR PRODUCT OF A AND B OUPTUT TO C

      INTEGER I,J,K,L,A1,A2,B1,B2
      REAL A(A1,A2),B(B1,B2),C(A1*B1,A2*B2)
      DO 10 I=1,A1
         DO 20 J=1,A2
            DO 30 K=1,B1
               DO 40 L=1,B2
                  C(K+(I-1)*B1,L+(J-1)*B2)=A(I,J)*B(K,L)
 40            CONTINUE
 30         CONTINUE
 20      CONTINUE
 10   CONTINUE
      RETURN
      END


      SUBROUTINE BNDEST(SEPM,PMU,PSIG,SEPMU,SEPSIG,CI)

*   CALCULATES THE FIRST AND SECOND ORDER VARIANCE ESTIMATES AND
*   CORRESPONDING CONFIDENCE INTERVAL SIZES

      INTEGER I,J
      REAL TEMP1(3,3),TEMP2(6,6),TEMP3(3,6),TEMP4(6,3),SSQMU(9),
     C     SSQSIG(36),SSQMSG(18),SSQSGM(18),SEPM(9,9),PSIGSQ(36,36),
     C     PSIG(6,6),SEPMU(3),SEPSIG(6),TEMP(36),PMUSQ(9,9),PMU(3,3),
     C     PMUSIG(18,18),PSIGMU(18,18),CI(2),STPMU(9),STPSIG(36)
     C     ,BND,BND1,BND2,BND3
      DO 10 I=1,3
         DO 20 J=1,3
            TEMP1(I,J)=SEPM(I,J)
 20      CONTINUE
         DO 30 J=1,6
            TEMP3(I,J)=SEPM(I,J+3)
 30      CONTINUE
 10   CONTINUE
      CALL STRING(TEMP1,SSQMU,3,3)
      CALL STRING(TEMP3,SSQMSG,3,6)
      DO 40 I=1,6
         DO 50 J=1,3
            TEMP4(I,J)=SEPM(I+3,J)
 50      CONTINUE
         DO 60 J=1,6
            TEMP2(I,J)=SEPM(I+3,J+3)
 60      CONTINUE
 40   CONTINUE
      CALL STRING(TEMP4,SSQSGM,6,3)
      CALL STRING(TEMP2,SSQSIG,6,6)
      CALL MXTMLT(PMU,SEPMU,TEMP,3,3,1)
      CALL MXTMLT(SEPMU,TEMP,BND,1,3,1)
      CALL MXTMLT(PSIG,SEPSIG,TEMP,6,6,1)
      CALL MXTMLT(SEPSIG,TEMP,BND1,1,6,1)
      BND1=BND1+BND
      CALL TENSR(PMU,PMU,PMUSQ,3,3,3,3)
      CALL MXTMLT(PMUSQ,SSQMU,TEMP,9,9,1)
      CALL MXTMLT(SSQMU,TEMP,BND,1,9,1)
      CALL TENSR(PSIG,PSIG,PSIGSQ,6,6,6,6)
      CALL MXTMLT(PSIGSQ,SSQSIG,TEMP,36,36,1)
      CALL MXTMLT(SSQSIG,TEMP,BND2,1,36,1)
      BND2=(BND+BND2)/2.0
      CALL TENSR(PMU,PSIG,PMUSIG,3,3,6,6)
      CALL MXTMLT(PMUSIG,SSQMSG,TEMP,18,18,1)
      CALL MXTMLT(SSQMSG,TEMP,BND,1,18,1)
      CALL TENSR(PSIG,PMU,PSIGMU,6,6,3,3)
      CALL MXTMLT(PSIGMU,SSQSGM,TEMP,18,18,1)
      CALL MXTMLT(SSQSGM,TEMP,BND3,1,18,1)
```

```
      BND2=BND2+((BND+BND3)/2.0)
      CALL STRING(PMU,STPMU,3,3)
      CALL STRING(PSIG,STPSIG,6,6)
      CALL MXTMLT(SSQMU,STPMU,BND,1,9,1)
      CALL MXTMLT(SSQSIG,STPSIG,BND3,1,36,1)
      BND2=BND2+(((BND+BND3)**2)/4.0)
      BND=BND1+BND2
      CI(1)=1.96*SQRT(BND1)
      CI(2)=1.96*SQRT(BND)
      RETURN
      END


      SUBROUTINE STRING(A1,A2,N1,N2)

*   STRINGS OUT THE MATRIX A1 AS A VECTOR A2

      REAL A1(N1,N2),A2(N1*N2)
      INTEGER I,N1,N2,J
      DO 10 I=1,N1
         DO 20 J=1,N2
            A2((I-1)*N2+J)=A1(I,J)
 20      CONTINUE
 10   CONTINUE
      RETURN
      END


      REAL FUNCTION HX(X,Y)
      REAL X,Y,SPHER(3),TEMP
      INCLUDE 'COM DEF'
      SPHER(1)=R*SIN(X)*COS(Y)-MU(1)
      SPHER(2)=R*SIN(X)*SIN(Y)-MU(2)
      SPHER(3)=R*COS(X)-MU(3)
      TEMP=-0.5*(((SPHER(1)**2)*SIGINV(1))+(2.0*SPHER(1)*SPHER(2)
     C              *SIGINV(2))+(2.0*SPHER(1)*SPHER(3)*SIGINV(3))
     C              +((SPHER(2)**2)*SIGINV(4))+(2.0*SPHER(2)*SPHER(3)
     C              *SIGINV(5))+((SPHER(3)**2)*SIGINV(6)))
      IF (TEMP .GT. -27.0 ) THEN
         HX=EXP(TEMP)/DENOM
      ELSE
         HX=0.0
      ENDIF
      RETURN
      END

      REAL FUNCTION CC1111(X,Y)
      EXTERNAL HX
      REAL X,Y
      CC1111=COS(X)*COS(Y)*HX(X,Y)
      RETURN
      END

      REAL FUNCTION CC1311(X,Y)
      EXTERNAL HX
      REAL X,Y
      CC1311=COS(3.0*X)*COS(Y)*HX(X,Y)
      RETURN
      END


      REAL FUNCTION CC3111(X,Y)
      EXTERNAL HX
      REAL X,Y
      CC3111=(COS(X)**3)*COS(Y)*HX(X,Y)
      RETURN
      END
```

```
REAL FUNCTION CS1111(X,Y)
EXTERNAL HX
REAL X,Y
CS1111=COS(X)*SIN(Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CS1311(X,Y)
EXTERNAL HX
REAL X,Y
CS1311=COS(3.0*X)*SIN(Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CS3111(X,Y)
EXTERNAL HX
REAL X,Y
CS3111=(COS(X)**3)*SIN(Y)*HX(X,Y)
RETURN
END

REAL FUNCTION SC1100(X,Y)
EXTERNAL HX
REAL X,Y
SC1100=SIN(X)*HX(X,Y)
RETURN
END

REAL FUNCTION SC1200(X,Y)
EXTERNAL HX
REAL X,Y
SC1200=SIN(2.0*X)*HX(X,Y)
RETURN
END

REAL FUNCTION SC1300(X,Y)
EXTERNAL HX
REAL X,Y
SC1300=SIN(3.0*X)*HX(X,Y)
RETURN
END


REAL FUNCTION SC2111(X,Y)
EXTERNAL HX
REAL X,Y
SC2111=(SIN(X)**2)*COS(Y)*HX(X,Y)
RETURN
END

REAL FUNCTION SC3121(X,Y)
EXTERNAL HX
REAL X,Y
SC3121=(SIN(X)**3)*(COS(Y)**2)*HX(X,Y)
RETURN
END

REAL FUNCTION SS2111(X,Y)
EXTERNAL HX
REAL X,Y
SS2111=(SIN(X)**2)*SIN(Y)*HX(X,Y)
RETURN
END
```

```
REAL FUNCTION SS3112(X,Y)
EXTERNAL HX
REAL X,Y
SS3112=(SIN(X)**3)*SIN(2.0*Y)*HX(X,Y)
RETURN
END

REAL FUNCTION SS3121(X,Y)
EXTERNAL HX
REAL X,Y
SS3121=(SIN(X)**3)*(SIN(Y)**2)*HX(X,Y)
RETURN
END

REAL FUNCTION CC1113(X,Y)
EXTERNAL HX
REAL X,Y
CC1113=COS(X)*COS(3.0*Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CC1131(X,Y)
EXTERNAL HX
REAL X,Y
CC1131=COS(X)*(COS(Y)**3)*HX(X,Y)
RETURN
END

REAL FUNCTION CC1211(X,Y)
EXTERNAL HX
REAL X,Y
CC1211=COS(2.0*X)*COS(Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CC1331(X,Y)
EXTERNAL HX
REAL X,Y
CC1331=COS(3.0*X)*(COS(Y)**3)*HX(X,Y)
RETURN
END

REAL FUNCTION CC1411(X,Y)
EXTERNAL HX
REAL X,Y
CC1411=COS(4.0*X)*COS(Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CC1511(X,Y)
EXTERNAL HX
REAL X,Y
CC1511=COS(5.0*X)*COS(Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CC1531(X,Y)
EXTERNAL HX
REAL X,Y
CC1531=COS(5.0*X)*(COS(Y)**3)*HX(X,Y)
RETURN
END
```

```
REAL FUNCTION CC2111(X,Y)
EXTERNAL HX
REAL X,Y
CC2111=(COS(X)**2)*COS(Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CC3113(X,Y)
EXTERNAL HX
REAL X,Y
CC3113=(COS(X)**3)*COS(3.0*Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CC3131(X,Y)
EXTERNAL HX
REAL X,Y
CC3131=(COS(X)**3)*(COS(Y)**3)*HX(X,Y)
RETURN
END

REAL FUNCTION CC5111(X,Y)
EXTERNAL HX
REAL X,Y
CC5111=(COS(X)**5)*COS(Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CC5113(X,Y)
EXTERNAL HX
REAL X,Y
CC5113=(COS(X)**5)*COS(3.0*Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CC5131(X,Y)
EXTERNAL HX
REAL X,Y
CC5131=(COS(X)**5)*(COS(Y)**3)*HX(X,Y)
RETURN
END

REAL FUNCTION CS1113(X,Y)
EXTERNAL HX
REAL X,Y
CS1113=COS(X)*SIN(3.0*Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CS1131(X,Y)
EXTERNAL HX
REAL X,Y
CS1131=COS(X)*(SIN(Y)**3)*HX(X,Y)
RETURN
END

REAL FUNCTION CS1211(X,Y)
EXTERNAL HX
REAL X,Y
CS1211=COS(2.0*X)*SIN(Y)*HX(X,Y)
RETURN
END
```

```
REAL FUNCTION CS1331(X,Y)
EXTERNAL HX
REAL X,Y
CS1331=COS(3.0*X)*(SIN(Y)**3)*HX(X,Y)
RETURN
END

REAL FUNCTION CS1411(X,Y)
EXTERNAL HX
REAL X,Y
CS1411=COS(4.0*X)*SIN(Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CS1511(X,Y)
EXTERNAL HX
REAL X,Y
CS1511=COS(5.0*X)*SIN(Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CS1531(X,Y)
EXTERNAL HX
REAL X,Y
CS1531=COS(5.0*X)*(SIN(Y)**3)*HX(X,Y)
RETURN
END

REAL FUNCTION CS2111(X,Y)
EXTERNAL HX
REAL X,Y
CS2111=(COS(X)**2)*SIN(Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CS3113(X,Y)
EXTERNAL HX
REAL X,Y
CS3113=(COS(X)**3)*SIN(3.0*Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CS3131(X,Y)
EXTERNAL HX
REAL X,Y
CS3131=(COS(X)**3)*(SIN(Y)**3)*HX(X,Y)
RETURN
END

REAL FUNCTION CS5111(X,Y)
EXTERNAL HX
REAL X,Y
CS5111=(COS(X)**5)*SIN(Y)*HX(X,Y)
RETURN
END

REAL FUNCTION CS5113(X,Y)
EXTERNAL HX
REAL X,Y
CS5113=(COS(X)**5)*SIN(3.0*Y)*HX(X,Y)
RETURN
END
```

```
REAL FUNCTION CS5131(X,Y)
EXTERNAL HX
REAL X,Y
CS5131=(COS(X)**5)*(SIN(Y)**3)*HX(X,Y)
RETURN
END

REAL FUNCTION SC1121(X,Y)
EXTERNAL HX
REAL X,Y
SC1121=SIN(X)*(COS(Y)**2)*HX(X,Y)
RETURN
END

REAL FUNCTION SC1221(X,Y)
EXTERNAL HX
REAL X,Y
SC1221=SIN(2.0*X)*(COS(Y)**2)*HX(X,Y)
RETURN
END

REAL FUNCTION SC1321(X,Y)
EXTERNAL HX
REAL X,Y
SC1321=SIN(3.0*X)*(COS(Y)**2)*HX(X,Y)
RETURN
END

REAL FUNCTION SC1400(X,Y)
EXTERNAL HX
REAL X,Y
SC1400=SIN(4.0*X)*HX(X,Y)
RETURN
END

REAL FUNCTION SC1421(X,Y)
EXTERNAL HX
REAL X,Y
SC1421=SIN(4.0*X)*(COS(Y)**2)*HX(X,Y)
RETURN
END

REAL FUNCTION SC1500(X,Y)
EXTERNAL HX
REAL X,Y
SC1500=SIN(5.0*X)*HX(X,Y)
RETURN
END

REAL FUNCTION SC1521(X,Y)
EXTERNAL HX
REAL X,Y
SC1521=SIN(5.0*X)*(COS(Y)**2)*HX(X,Y)
RETURN
END

REAL FUNCTION SC3100(X,Y)
EXTERNAL HX
REAL X,Y
SC3100=(SIN(X)**3)*HX(X,Y)
RETURN
END
```

```
REAL FUNCTION SC4111(X,Y)
EXTERNAL HX
REAL X,Y
SC4111=(SIN(X)**4)*COS(Y)*HX(X,Y)
RETURN
END

REAL FUNCTION SC4113(X,Y)
EXTERNAL HX
REAL X,Y
SC4113=(SIN(X)**4)*COS(3.0*Y)*HX(X,Y)
RETURN
END

REAL FUNCTION SC4131(X,Y)
EXTERNAL HX
REAL X,Y
SC4131=(SIN(X)**4)*(COS(Y)**3)*HX(X,Y)
RETURN
END

REAL FUNCTION SC5112(X,Y)
EXTERNAL HX
REAL X,Y
SC5112=(SIN(X)**5)*COS(2.0*Y)*HX(X,Y)
RETURN
END

REAL FUNCTION SC5114(X,Y)
EXTERNAL HX
REAL X,Y
SC5114=(SIN(X)**5)*COS(4.0*Y)*HX(X,Y)
RETURN
END

REAL FUNCTION SC5121(X,Y)
EXTERNAL HX
REAL X,Y
SC5121=(SIN(X)**5)*(COS(Y)**2)*HX(X,Y)
RETURN
END

REAL FUNCTION SC5141(X,Y)
EXTERNAL HX
REAL X,Y
SC5141=(SIN(X)**5)*(COS(Y)**4)*HX(X,Y)
RETURN
END

REAL FUNCTION SC1212(X,Y)
EXTERNAL HX
REAL X,Y
SC1212=SIN(2.0*X)*COS(2.0*Y)*HX(X,Y)
RETURN
END

REAL FUNCTION SC1412(X,Y)
EXTERNAL HX
REAL X,Y
SC1412=SIN(4.0*X)*COS(2.0*Y)*HX(X,Y)
RETURN
END
```

```
REAL FUNCTION SS1112(X,Y)
EXTERNAL HX
REAL X,Y
SS1112=SIN(X)*SIN(2.0*Y)*HX(X,Y)
RETURN
END

REAL FUNCTION SS1121(X,Y)
EXTERNAL HX
REAL X,Y
SS1121=SIN(X)*(SIN(Y)**2)*HX(X,Y)
RETURN
END

REAL FUNCTION SS1212(X,Y)
EXTERNAL HX
REAL X,Y
SS1212=SIN(2.0*X)*SIN(2.0*Y)*HX(X,Y)
RETURN
END

REAL FUNCTION SS1221(X,Y)
EXTERNAL HX
REAL X,Y
SS1221=SIN(2.0*X)*(SIN(Y)**2)*HX(X,Y)
RETURN
END

REAL FUNCTION SS1312(X,Y)
EXTERNAL HX
REAL X,Y
SS1312=SIN(3.0*X)*SIN(2.0*Y)*HX(X,Y)
RETURN
END

REAL FUNCTION SS1321(X,Y)
EXTERNAL HX
REAL X,Y
SS1321=SIN(3.0*X)*(SIN(Y)**2)*HX(X,Y)
RETURN
END

REAL FUNCTION SS1412(X,Y)
EXTERNAL HX
REAL X,Y
SS1412=SIN(4.0*X)*SIN(2.0*Y)*HX(X,Y)
RETURN
END

REAL FUNCTION SS1421(X,Y)
EXTERNAL HX
REAL X,Y
SS1421=SIN(4.0*X)*(SIN(Y)**2)*HX(X,Y)
RETURN
END

REAL FUNCTION SS1512(X,Y)
EXTERNAL HX
REAL X,Y
SS1512=SIN(5.0*X)*SIN(2.0*Y)*HX(X,Y)
RETURN
END
```

```
      REAL FUNCTION SS1521(X,Y)
      EXTERNAL HX
      REAL X,Y
      SS1521=SIN(5.0*X)*(SIN(Y)**2)*HX(X,Y)
      RETURN
      END

      REAL FUNCTION SS4111(X,Y)
      EXTERNAL HX
      REAL X,Y
      SS4111=(SIN(X)**4)*SIN(Y)*HX(X,Y)
      RETURN
      END

      REAL FUNCTION SS4113(X,Y)
      EXTERNAL HX
      REAL X,Y
      SS4113=(SIN(X)**4)*SIN(3.0*Y)*HX(X,Y)
      RETURN
      END

      REAL FUNCTION SS4131(X,Y)
      EXTERNAL HX
      REAL X,Y
      SS4131=(SIN(X)**4)*(SIN(Y)**3)*HX(X,Y)
      RETURN
      END

      REAL FUNCTION SS5112(X,Y)
      EXTERNAL HX
      REAL X,Y
      SS5112=(SIN(X)**5)*SIN(2.0*Y)*HX(X,Y)
      RETURN
      END

      REAL FUNCTION SS5114(X,Y)
      EXTERNAL HX
      REAL X,Y
      SS5114=(SIN(X)**5)*SIN(4.0*Y)*HX(X,Y)
      RETURN
      END

      REAL FUNCTION SS5121(X,Y)
      EXTERNAL HX
      REAL X,Y
      SS5121=(SIN(X)**5)*(SIN(Y)**2)*HX(X,Y)
      RETURN
      END

      REAL FUNCTION SS5141(X,Y)
      EXTERNAL HX
      REAL X,Y
      SS5141=(SIN(X)**5)*(SIN(Y)**4)*HX(X,Y)
      RETURN
      END


      COM DEF
      REAL R,SIG(6),SIGINV(6),DET,DENOM,MU(3),PI,A,B,ERRABS,ERRREL
     C     ,MSIG(3,3),MUSET(3),SIGSET(3,3),SEPSET
      INTEGER IRULE,ISEED,NUMTR
      COMMON/COM1/R,SIG,SIGINV,DET,DENOM,MU,ERRABS,ERRREL,MSIG,
     C      MUSET,SIGSET,SEPSET,IRULE,NUMTR,ISEED
      PARAMETER(PI=3.1415927)
```

```
WVEC DEF
REAL W(12),CC(17),CS(18),SC(19),SS(18)
COMMON/WCC/W,CC,CS,SC,SS


PMUCOM DEF
REAL S(9,6),ST(6,9),NUMSIM
COMMON/COM3/S,ST,NUMSIM


SEPCOM DEF
REAL CXX,CYY,CZZ,SQPIINV,QUPITW,CSANT,C,RADIUS,MN(3)
COMMON/COM2/CXX,CYY,CZZ,CSANT,C,RADIUS,MN
PARAMETER(SQPIINV=0.5641896,QUPITW=0.05626977)
```

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c.1 Second order approximation for variance of SEP.